Degenerations of Del Pezzo Surfaces and Gromov-Witten Invariants of the Hilbert Scheme of Conics

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Abstract: This paper investigates low-codimension degenerations of Del Pezzo surfaces. As an application we determine certain characteristic numbers of Del Pezzo surfaces. Finally, we analyze the relation between the enumerative geometry of Del Pezzo surfaces and the Gromov-Witten invariants of the Hilbert scheme of conics in $\mathbb{P}^N$.

Contents

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1 Introduction:

This paper investigates the degenerations of Del Pezzo surfaces D_n embedded in $\mathbb{P}^N$ by their anti-canonical bundle. Due to the vast number of possibilities, we restrict our attention to describing simple specializations of D_n. As an application we determine some characteristic numbers of Del Pezzo surfaces. Finally, we discuss the relation between these numbers and the Gromov-Witten invariants of the Hilbert scheme of conics. We work exclusively over the complex number field $\mathbb{C}$.

This is a sequel to [C] where we studied the enumerative geometry of rational normal surface scrolls. Already in that case, to obtain recursive formulae for the number of surfaces incident to general linear spaces, we needed to impose strong non-degeneracy assumptions by requiring enough of the linear spaces to be points. The case of Del Pezzo surfaces is more complicated, but instructive to consider. The new features of this case can be summarized as follows:

1. Reducible surfaces that are limits of one-parameter families of scrolls are again unions of scrolls. Del Pezzo surfaces exhibit a much larger variety of degenerations (§3). For example, a Del Pezzo surface can degenerate to a union of scrolls, a union of a Veronese surface and a scroll, a union of a Del Pezzo surface of lower degree and planes, a union of a rational cone and an elliptic cone. This partial list indicates that we cannot hope for a reasonable recursive formula for characteristic numbers of Del Pezzo surfaces via degeneration methods except in very special cases.

2. The hyperplane sections of Del Pezzo surfaces are not rational, but elliptic curves. The case of genus one curves in $\mathbb{P}^N$ is the last case where we have a firm understanding of the enumerative geometry of curves satisfying incidences with linear spaces ([V]). Consequently, the Del Pezzo surfaces lie at the perimeter of surfaces whose enumerative geometry we can analyze via degenerations.

3. Unlike scrolls Del Pezzo surfaces can have non-trivial moduli. This often makes it challenging to recognize limits.

An interesting observation resulting from our investigations is that degenerations of higher dimensional varieties exhibit qualitative behavior fundamentally different from that of curves. Degenerations of incidence and tangency conditions on curves with respect to linear spaces result in a closed system of enumerative problems. The limits of curves again satisfy similar conditions. The limits of surfaces, on the other hand, can be subject to arbitrarily complicated conditions. Since degeneration arguments are prevalent in algebraic geometry, this crucial difference is important to note.

We now sketch an outline of the paper.

Notation. Let S^* be the dual of the tautological bundle over $\mathbb{G}(2, N)$, the Grassmannian of planes in $\mathbb{P}^N$. Let X^N denote $\mathbb{P}(\mathrm{Sym}^2 S^*)$. X^N is a projective

bundle over $\mathbb{G}(2, N)$. We can interpret it as the space of pairs of a plane and a conic in the plane.

The limits. We produce a list of potential non-degenerate limits of D_n that can occur in one-parameter families (§3) using a classical theorem of Del Pezzo and Nagata (§2.3), which classifies surfaces of degree n in $\mathbb{P}^n$. We then exhibit families realizing the degenerations of D_n relevant to our counting problems and we describe the limiting positions of geometrically significant curves.

We use two techniques to construct families of D_n specializing to a given limit. We specialize the base points of the linear system of cubics on $\mathbb{P}^2$ in various ways to obtain classical constructions. More interestingly, since Del Pezzo surfaces D_n are ruled by conics, we can interpret them as curves in X^N. Let d_n denote the cohomology class of a curve in X^N arising from a one-parameter family of conics on D_n. Given a potential limit surface, we can try to find a curve C of conics in the class d_n which sweeps it. If we can deform C to a curve of conics arising from a smooth D_n, then we can conclude that the surface arises as a degeneration of D_n.

Example. For instance, it takes ingenuity to find a specialization of the base points in order to obtain a family of D_n ($n < 8$) degenerating to the projection of the rational scroll $S_{2,n-2}$ from a point on the plane of a conic on $S_{2,n-2}$. However, it is easy to see that the curve of reducible conics consisting of a fiber line and the double line deforms to a curve of conics on D_n (see §3).

Characteristic numbers. By the *characteristic number problem* we mean the problem of computing the number of varieties of a given type that meet the 'appropriate' number of linear spaces in general position. Classically characteristic numbers also allow tangency conditions; however, in this paper we will consider only incidence conditions. Using our description of the degenerations of D_n we determine some characteristic numbers of D_3 and D_4. Although most of these numbers can also be obtained by classical methods, our method has the advantage of circumventing tedious cohomology calculations and yields numbers of surfaces satisfying divisorial conditions, which are hard to obtain classically. We can also determine a few of the characteristic numbers of D_5. However, for $n \geq 6$ and essentially for $n = 5$, the degenerations get too complicated for the method to terminate and give actual numbers. If instead we ask for the number of D_n containing a fixed degree n elliptic normal curve and satisfying incidences with linear spaces, then the degeneration method gives a few more answers for $n = 5$ and 6. I do not know of a classical method to compute these numbers when $n > 4$.

The method. In order to count surfaces incident to various linear spaces, we degenerate the linear spaces one by one to a hyperplane H until we force any surface meeting them to become reducible. If we have enough point conditions to satisfy our non-degeneracy assumptions, we know the possible reducible surfaces that occur in the limit. We can hope to count surfaces by further breaking each of the pieces of the limit surface to obtain simpler surfaces. This hope is in general upset by the appearance of singular surfaces and more and more complicated conditions on hyperplane sections of the surfaces. However, the

method still works in many cases (see examples in §4) and with some effort should extend to more cases than covered here.

Gromov-Witten invariants of X^N. It is natural to ask for the relation between the enumerative numbers for D_n and the Gromov-Witten invariants of X^N. X^N is not a convex variety (§2.2), i.e. there are maps $f : \mathbb{P}^1 \to X^N$ for which $h^1(\mathbb{P}^1, f^*T_{X^N}) \neq 0$. Its Kontsevich spaces of genus zero stable maps often have components of more than the expected dimension.

When a variety V is homogeneous (in particular convex), then the Gromov-Witten invariants count the number of curves that meet general subvarieties of V. However, when the variety is not convex, there can be virtual contributions to the Gromov-Witten invariants. It is usually a hard problem to decide when the Gromov-Witten invariants of a non-convex space are enumerative. Gathmann [Ga] and Göttsche and Pandharipande [GP] discuss this problem for the blow-ups of $\mathbb{P}^N$ and $\mathbb{P}^2$, respectively.

In general the Gromov-Witten invariants of X^N for the class d_n are not enumerative (§6). However, we prove that the Gromov-Witten invariants involving incidences to linear spaces are enumerative when $n = 3$ and when $n = 4$ provided that there are not any $\mathbb{P}^3$s incident to all the linear spaces. As a corollary we compute some Gromov-Witten invariants of X^N.

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2 Preliminaries

2.1 Del Pezzo surfaces

In this subsection we discuss the basic geometry of Del Pezzo surfaces. For more details consult [Bv] Ch. 4, [GH] §1 Ch. 4 or [Fr] Ch. 5.

Del Pezzo surfaces are smooth complex surfaces with ample anti-canonical bundle $-K$. Except for $\mathbb{P}^1 \times \mathbb{P}^1$, they can be realized as the blow-up of $\mathbb{P}^2$ in fewer than 9 points no three of which lie on a line and no six of which lie on a conic. To have a more uniform discussion we exclude $\mathbb{P}^1 \times \mathbb{P}^1$. We denote Del Pezzo surfaces by D_n where n is the degree K^2 of the anti-canonical bundle. Equivalently, D_n is the blow-up of $\mathbb{P}^2$ in $9 - n$ general points p_i. The anti-canonical series $|-K|$ on a Del Pezzo surface can be interpreted as the linear series of cubics on $\mathbb{P}^2$ having p_i as base points, therefore

$$h^0(D_n, -K) = 10 - n.$$

We limit our discussion to Del Pezzo surfaces D_n embedded in $\mathbb{P}^n$ by their anti-canonical bundle, i.e. to D_n with $n \geq 3$.

Geometric description. These surfaces display a rich geometry and often have nice determinantal descriptions. D_3 is a cubic surface in $\mathbb{P}^3$. D_4 is the complete intersection of two quadric threefolds in $\mathbb{P}^4$. D_5 is a fourfold hyperplane section of the Grassmannian $\mathbb{G}(1,4)$ under its Plücker embedding. D_6 is a two fold hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ or it is the hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Finally, D_9 is the cubic Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$.

The Picard group of D_n is isomorphic to $\mathbb{Z}^{10-n}$ generated by the classes H, the pull back of the hyperplane class from $\mathbb{P}^2$, and E_i, $1 \leq i \leq 9-n$, the exceptional divisors of the blow-up. The intersection pairing is

$$H^2 = 1, \qquad H \cdot E_i = 0, \qquad E_i \cdot E_j = -\delta_{i,j}.$$

In terms of these classes the anti-canonical class is $-K = 3H - \sum_{i=1}^{9-n} E_i$.

By Bertini's theorem a general hyperplane section of D_n is a smooth elliptic curve of degree n. These curves are projectively normal.

Curves on D_n. During the degenerations it is important to know the limits of lines, conics and hyperplane sections on D_n. On D_n the effective curve classes containing an irreducible curve of a given arithmetic genus g and degree d are easy to determine. We can express the class of any curve as $aH - \sum_{i=1}^{9-n} b_i E_i$. Since the surface is embedded by $|-K|$, the degree condition implies that

$$3a - \sum_{i=1}^{9-n} b_i = d.$$

The genus formula translates to

$$a^2 - \sum_{i=1}^{9-n} b_i^2 = 2g - 2 + d.$$

Using the Cauchy-Schwarz inequlity

$$\left(\sum_{i=1}^{9-n} b_i\right)^2 \leq (9-n) \sum_{i=1}^{9-n} b_i^2,$$

we find the choices for a and then solve for the b_i satisfying the two equations. In the rest of the paper we will use this scheme to determine curve classes without further mention. For the convenience of the reader we enumerate the classes of lines and conics on D_n.

Lemma 2.1 *On the Del Pezzo surfaces D_n ($n \geq 3$) the classes of lines are E_i, $1 \leq i \leq 9-n$, $H - E_i - E_j$, $i \neq j$ and $2H - E_a - E_b - E_c - E_d - E_e$ where a,b,c,d,e are distinct, whenever these classes exist.*

Lemma 2.2 *On the Del Pezzo surfaces D_n ($n \geq 3$) the classes of conics are $H - E_i$, $2H - E_a - E_b - E_c - E_d$, $3H - 2E_a - E_b - E_c - E_d - E_e - E_f$ where a,b,c,d,e,f are distinct, whenever these classes exist.*

Moduli of Del Pezzo surfaces. The surfaces D_3 and D_4 have a four and two dimensional moduli space, respectively. We will not be concerned with the construction or properties of these moduli spaces.

Singular Del Pezzo surfaces. A *singular Del Pezzo surface* $D_n^{(s)}$ is an irreducible surface of degree n in $\mathbb{P}^n$ which has isolated double points. $D_n^{(s)}$ is also the image of the blow-up of $\mathbb{P}^2$ in $9-n$ points. $D_n^{(s)}$ arises when the points we blow up to obtain D_n become infinitely near, fail to be in general linear position or lie on a conic (when $n = 3$).

The list of the combinations of double points that occur on $D_n^{(s)}$ is long. When $n = 3$, Bruce and Wall give a very nice description [BW]. The type and combination of the double points that occur on a cubic surface are all obtained by deleting vertices (and the edges adjacent to them) from the extended E_6 diagram.

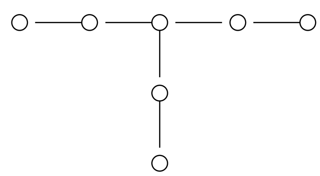

Figure 1: The extended E_6 diagram

Conversely, every combination of Du Val singularities that arises by deleting vertices from the extended E_6 diagram occurs on some cubic surface.

Tangent planes to D_n along curves. When studying limiting positions of hyperplane sections in one-parameter families of surfaces, it is essential to have estimates on the dimension of the space of hyperplanes tangent to the components of the limit surface along common curves. The dimension of the space of hyperplanes tangent to a smooth $D_n \subset \mathbb{P}^n$ along a line is $\max(-1, n-5)$. However, if the surface has a double point this estimate can change. For example, on a smooth $D_4 \subset \mathbb{P}^4$ there are not any hyperplanes tangent to the surface everywhere along a line; however, when the surface acquires an ordinary double point, there can be one. Consequently, one has to exercise caution to consider all possible singularities when giving estimates. For simplicity, we will often exclude singular surfaces from the discussion.

The dimension of the space of Del Pezzo surfaces in $\mathbb{P}^N$. If we fix linear spaces Λ^{a_i} of dimension a_i in general position such that

$$\sum_i (N - 2 - a_i) = N(n+1) - n + 10,$$

then there will be finitely many smooth D_n meeting all the linear spaces by the following dimension count.

Lemma 2.3 *The dimension of the locus in the Hilbert scheme whose general point corresponds to a smooth D_n in $\mathbb{P}^N$ is*

$$N(n+1) - n + 10.$$

Proof: Realize D_n as the image of a map from the blow-up of $\mathbb{P}^2$ at $9-n$ points by choosing $N+1$ sections in $|-K|$ and projectivizing. We need to add the dimension of the moduli space or subtract the dimension of the automorphism group which amounts to adding $10-2n$. $\square$

We will determine the number of D_n in some cases using degenerations. For future reference we recall the following well-known fact (see [V] §5).

Lemma 2.4 *The dimension of the component of the Hilbert scheme whose general point corresponds to a smooth elliptic curve of degree $n+1$ spanning a $\mathbb{P}^n$ in $\mathbb{P}^N$ is*

$$(N-n)(n+1) + (n+1)^2.$$

Rational Scrolls. During the degenerations of D_n we will encounter rational surface scrolls. We refer the reader to [C] for a detailed discussion of their geometry.

A rational normal scroll $S_{k,l}$ is abstractly the Hirzebruch surface F_{l-k} embedded in $\mathbb{P}^{k+l+1}$ by the complete linear series $e + lf$, where e, f are the usual generators of the Picard group of F_{l-k} satisfying $e^2 = k - l,\ f^2 = 0,\ e \cdot f = 1$. The classes e, f are the classes of the exceptional curve E and of a fiber F, respectively. The surface can be explicitly constructed by taking a rational normal k curve and a rational normal l curve with disjoint linear spans; choosing an isomorphism between the curves; and taking the union of lines joining corresponding points.

We will refer to a curve class $e + mf$ as a **section class** and to a curve class $2e + mf$ as a **bisection class**. Irreducible curves in section and bisection classes are sections and bisections of the projective bundle over $\mathbb{P}^1$, respectively. In addition to the cohomology calculations in §2 of [C], we will use

$$h^0(F_r, \mathcal{O}_{F_r}(2e + (r+2)f)) = \begin{cases} 9 & : \quad r \leq 2 \\ h^0(F_r, \mathcal{O}_{F_r}(e + (r+2)f)) & : \quad r \geq 3 \end{cases}$$

which follows by considering the exact sequences

$$\begin{aligned} 0 \to \mathcal{O}_{F_r}(2e + mf) \to \mathcal{O}_{F_r}(2e + (m+1)f) &\to \mathcal{O}_F(2) \to 0 \\ 0 \to \mathcal{O}_{F_r}(e + mf) \to \mathcal{O}_{F_r}(2e + mf) \ &\to \mathcal{O}_E(m - 2r) \to 0. \end{aligned}$$

The Veronese surface. During the degenerations we will also encounter the Veronese surface, the image of $\mathbb{P}^2$ in $\mathbb{P}^5$ given by the complete linear system of conics. A central fact is that the Veronese surface together with the rational normal scrolls are the only non-degenerate irreducible surfaces of degree $n-1$ in $\mathbb{P}^n$ ([GH] p. 525).

2.2 The Geometry of the Space of Conics

Since we will rely on the description of D_n as a curve of conics, we recall the basic facts about the Hilbert scheme of conics in $\mathbb{P}^N$.

Let S^* denote the dual of the tautological bundle on the Grassmannian $\mathbb{G}(2, N)$ of planes in $\mathbb{P}^N$. Recall that X^N was defined to be $\mathbb{P}(\mathrm{Sym}^2 S^*)$. Let

$$\pi : X^N \to \mathbb{G}(2, N)$$

be the natural projection map.

The Chow ring of X^N is generated by the pull-back of the classes on $\mathbb{G}(2, N)$ and the first chern class of the tautological bundle on X^N ([Ful] 8.3.4). The Picard group has 2 generators which we can express in terms of geometric cycles. Let ω be the class of conics which meet a fixed $\mathbb{P}^{N-2}$ (i.e. the pull-back of the hyperplane class of $\mathbb{P}^N$ by the natural morphism) and η the class of conics whose planes meet a fixed $\mathbb{P}^{N-3}$ (more precisely, the first chern class of the tautological bundle on X^N). The locus of reducible conics Δ, whose class we denote by δ, is also a divisor. The intersection products below (see Table 1) imply that $\delta = 3\omega - 4\eta$.

Similarly, there are two natural curve classes on X^N. Let a be the class of conics that lie in a fixed plane and pass through 4 general points on the plane. Let b be the class of conics that are cut out on a fixed quadric surface by a pencil of hyperplanes. Each conic class contains a one-parameter family of disjoint conics on D_n, $n \neq 9$, so gives rise to a curve in X^N. For example, D_3 gives rise to 27 rational curves in X^3. Let d_n denote the class in X^N of the curve of conics on D_n in a fixed cohomology class. Table 1 below implies that

$$d_n = (4 - n)a + (n - 2)b.$$

$\cdot$	ω	η	δ
a	1	0	3
b	2	1	2
d_n	n	$n-2$	$8-n$

Table 1: Intersection Products

For $\beta \in H^2(X^N, \mathbb{Z})$ let $\overline{M}_{0,m}(X^N, \beta)$ denote the Kontsevich space of stable maps in the class β. In general the Kontsevich spaces of X^N can have components of larger than expected dimension. One of the simplest examples is $\overline{M}_{0,0}(X^N, -da + db)$ for $d > 1$. Using the Euler sequence for the tangent bundle ([Ful] 3.2.11) one can check that the canonical bundle of X^N is given by

$$K_{X^N} = -6\omega + (7 - N)\eta.$$

Hence, the expected dimension of $\overline{M}_{0,0}(X^N, -da + db)$ is

$$\dim\ X^N + c_1(X)(-da + db) - 3 = Nd - d + 3N - 4.$$

Take a cone C over a rational curve of degree d and a line l not necessarily contained in C, but meeting it at the vertex. The curve in X^N whose points correspond to the union of l with a line of C has class $-da+db$. The dimension of cone and line pairs is $Nd+3N-5$. So when $d>1$, this provides us with a Kontsevich space of the "wrong" dimension. We will see more examples in §6. For future reference we note that the expected dimension of $\overline{M}_{0,m}(X^N, d_n)$ is

$$N(n+1)-n+10+m.$$

When $m=0$, this agrees with the dimension in Lemma 2.3.

2.3 The Classification of degree n surfaces in $\mathbb{P}^n$.

In this subsection we state the classification theorem for reduced, irreducible, non-degenerate surfaces of degree n in $\mathbb{P}^n$. This is a classical theorem of Del Pezzo and Nagata whose proof can be found in [Na].

Theorem 2.5 *An irreducible, reduced, non-degenerate surface of degree n in $\mathbb{P}^n$ is one of the following:*

1. *A projection to $\mathbb{P}^n$ of a scroll of degree n in $\mathbb{P}^{n+1}$,*
2. *A projection to $\mathbb{P}^4$ of the Veronese surface in $\mathbb{P}^5$,*
3. *A Del Pezzo surface, possibly with finitely many isolated double points,*
4. *The image of F_0 or F_2 in $\mathbb{P}^8$ given by their anti-canonical map,*
5. *A cone over an elliptic curve of degree n in $\mathbb{P}^{n-1}$.*

3 The limits of Del Pezzo surfaces

In this section we describe the non-degenerate surfaces in $\mathbb{P}^n$ that can arise as limits of D_n. We appeal to the description of D_n as a curve in X^n. Since D_9 does not contain any conics, we restrict the values of n to $3 \leq n \leq 8$.

3.1 Constraints on the Degenerations

Notation. Let $f : \mathcal{Y} \to B$ be a flat family of surfaces in $\mathbb{P}^N$ over a smooth, connected curve. Let $b_0 \in B$ be a marked point and let $\mathcal{Y}_0$ denote the fiber of f over b_0. We assume that $\mathcal{Y}_b$ for points $b \neq b_0$ is a Del Pezzo surface D_n. We also assume that $\mathcal{Y}_0$ spans a $\mathbb{P}^n$ and its components are reduced. We preserve the notation we used in §2 for X^N.

Adjacent components. Let Y be a reducible surface connected in codimension 1. We refer to two components that share a common curve as *adjacent components.* The *dual graph* of the surface consists of a vertex for each irreducible component and an edge between adjacent ones.

Lemma 3.1 *$\mathcal{Y}_0$ is a surface of degree n whose components are ruled by lines or conics. There exists a connected subgraph containing all the vertices of its dual graph such that adjacent components share a common line or conic.*

Proof: Each D_n in the family gives rise to a collection of rational curves in X^N. After a finite base change totally ramified over b_0, we can select a conic class on each surface away from b_0. We denote the new family by $\mathcal{Y}' \to B'$. This family induces a curve in the Kontsevich space of stable maps $\overline{M}_{0,0}(X^N, d_n)$. The limit of the family in $\overline{M}_{0,0}(X^N, d_n)$ is a map from a tree of rational curves to X^N. The restriction of the universal curve over X^N to the family of curves maps to $\mathbb{P}^N$ giving rise to a family of surfaces which agrees with $\mathcal{Y}'$ except possibly over b_0'. There is a scheme structure on the limit surface which makes the family flat. Since over a smooth curve there is a unique way to complete a family to a flat family, this family agrees with our original family. We conclude that the components of $\mathcal{Y}_0$ are ruled by conics or lines. The last assertion is clear. $\square$

Proposition 3.2 *Each component of $\mathcal{Y}_0$ is one of the following:*

1. *A Veronese surface in $\mathbb{P}^5$ or a non-degenerate scroll of degree k in $\mathbb{P}^{k+1}$,*
2. *A projection of one of the surfaces in 1 to a surface with a double line in a one dimensional lower projective space,*
3. *A cone over an elliptic curve of degree k in $\mathbb{P}^{k-1}$, or*
4. *A Del Pezzo surface D_n, possibly with isolated double points.*

Proof: If $\mathcal{Y}_0$ is irreducible, then it is a non-degenerate surface of degree n in $\mathbb{P}^n$. $\mathcal{Y}_0$ cannot be the anti-canonical image of F_0 or F_2 since these surfaces do not contain any lines. The flat limit of the lines in the family of D_8 would be a line. Similarly, $\mathcal{Y}_0$ cannot be the projection of a rational scroll or Veronese surface with isolated singularities. The hyperplane section of such a surface has arithmetic genus 0 instead of 1. By Theorem 2.5 we conclude that $\mathcal{Y}_0$ is one of the surfaces in cases 2, 3 or 4.

Two components. Suppose $\mathcal{Y}_0$ has two irreducible components W and Z of degrees d_W and d_Z. By Lemma 3.1 they share a line or a conic.

Suppose W and Z meet in a conic (possibly reducible or non-reduced). Then the linear spaces they span contain a common plane, so their total span is at most $\mathbb{P}^{d_Y+d_Z}$. We conclude that the surfaces must be minimal degree surfaces, so they are one of the surfaces in case 1. Since the Veronese surface does not contain any lines, at most one of the surfaces can be a Veronese surface. Therefore, the surfaces are either two scrolls meeting along a conic or a Veronese and a scroll meeting along a conic.

Suppose W and Z meet along a line. If both of the surfaces are minimal degree surfaces and meet generically transversely along the line, then their union cannot be a limit of Del Pezzo surfaces since their hyperplane sections have arithmetic genus 0 instead of 1. It is possible for two rational cones tangent along a line to be a limit of D_n, but this is included in the previous case.

We can, therefore, assume that W spans only a $\mathbb{P}^{d_W}$. Z must span a $\mathbb{P}^{d_Z+1}$ and meet W along the line generically transversely. Since a Veronese surface does not contain any lines, Z is a rational normal scroll. Theorem 2.5 implies that W is a Del Pezzo surface possibly with finitely many double points, a cone over an elliptic curve or the projection of a scroll or a Veronese. Moreover, if W

is the projection of a scroll or the Veronese, it must have a double line because otherwise a general hyperplane section would have arithmetic genus 0.

Further constraints. If W is a Del Pezzo surface and Z is not a plane, then both W and Z must be singular. Suppose to the contrary that W is a smooth D_k and Z is a scroll of degree greater than 1 meeting it in a line l_C. The limit of a curve of conics on D_n is a connected curve of conics on $\mathcal{Y}_0$. Since W and Z do not have a conic in common, W is not ruled by lines and the double of l_C is not a conic class on W, the conics on $\mathcal{Y}_0$ must consist of a curve of conics on W union a curve of lines on Z together with a fixed line l_F on W intersecting each line on Z. Hence, Z must be a cone. The fixed line l_F cannot be l_C, so the conic class on W is the conic class $[l_F]+[l_C]$. The intersection of the variety of reducible conics with this curve has more than $8-n$ isolated points (see table in §2), hence the curve cannot be deformed to a smooth curve in the class d_n. We conclude that W is also singular. Furthermore, the same argument shows that as the degree of the scroll increases, the residual Del Pezzo surface is forced to have worse singularities. For example, if D_n breaks into a cubic cone union D_{n-3}, then D_{n-3} must have a singularity worse than an ordinary double point. Using the list of singularities and the number of lines on the singular surfaces it is not too hard to make a list of possibilities.

Remark. In case W is not D_3 or D_4 the previous constraint follows by an elementary dimension count. However, since D_3 and D_4 have moduli the dimension count only shows that W cannot be a general smooth D_3 or D_4.

By a similar argument if W is a cone over an elliptic curve, then Z is a rational cone with matching vertex. The limit curve of conics has to be reducible. One component of the curve must consist of line pairs on W joining the points identified by the hyperelliptic involution on the elliptic curve. The other component must consist of a fixed line in W union lines in Z. The claim follows.

More components. Now we allow $\mathcal{Y}_0$ to have more than two components.

Observation. If a subsurface S of $\mathcal{Y}_0$ of degree d spans exactly $\mathbb{P}^d$, then all the remaining components are minimal degree scrolls meeting the components adjacent to them in lines. The components adjacent to S need to be attached to S along at least a line. To have the resulting surface be non-degenerate, the surface we attach must have maximal possible span for its degree and meet at most one of the components of S in a line. The observation follows by induction.

• Suppose $\mathcal{Y}_0$ contains three components U_i of degree d_i pairwise meeting in distinct curves. U_i spans at most $\mathbb{P}^{d_i+1}$. The three components together span at most a linear space of dimension $\sum_i d_i$ with equality if and only if each of the surfaces have maximal span and their common curves are concurrent lines. By the observation we conclude that each component is a scroll and the U_i each contain a pair of intersecting lines.

• Suppose $\mathcal{Y}_0$ contains two components meeting in a conic, possibly reducible or non-reduced. Then the two components are either two scrolls meeting along a conic or a Veronese surface and a scroll meeting along a conic. By the observation all the other components must be scrolls.

• We can assume that all components of $\mathcal{Y}_0$ meet pairwise in lines and no three components meet pairwise in distinct curves. Suppose one of the components U of degree d spans $\mathbb{P}^d$. By Theorem 2.5 and the argument given for the case when $\mathcal{Y}_0$ has two components, U is the projection of a Veronese surface or a rational scroll with a double line, a cone over an elliptic normal curve or a Del Pezzo surface possibly with finitely many singularities. By the observation all the other components are scrolls. By an argument similar to the two component case we can deduce that if V is a smooth Del Pezzo surface then all the adjacent scrolls are planes and they are joined to V along non-intersecting lines. In case V is a cone over an elliptic normal curve, the adjacent components are rational cones whose vertices coincide with the vertex of V.

• Finally, we can assume that all the components are rational scrolls meeting pairwise in lines. The hyperplane sections of the surface ought to have arithmetic genus 1. This concludes the description of the components. $\square$

Corollary 3.3 *1. At most one component of $\mathcal{Y}_0$ is a Veronese surface. If a component is a Veronese surface, all other components are rational normal scrolls.*

2. If a component U of $\mathcal{Y}_0$ of degree d spans only $\mathbb{P}^d$, then all the other components are rational normal scrolls. If U is a smooth Del Pezzo surface or a cone over an elliptic normal curve, the scrolls have to satisfy the constraints described in the proof of Proposition 3.2.

3.2 Explicit Families Realizing the Degenerations

One might hope that few of the surfaces described in Proposition 3.2 actually occur as components of the limtis of D_n. Unfortunately this is not so. We now give examples of families realizing most of the irreducible or two-component possibilities listed in Proposition 3.2. We assume $n < 8$ throughout.

Notation. Let A denote a disk in $\mathbb{C}$. Let $a_0 \in A$ denote a marked point in the disk. We denote sections of a family of varieties over A by s_i. Finally, let $N_{X/Y}$ denote the normal bundle of X in Y.

I. The Del Pezzo surfaces with isolated double points all arise by specializing the base points of the linear system of cubics on $\mathbb{P}^2$. From this description one can determine the limits of the lines and conics.

For example, to construct a family of cubic surfaces specializing to a cubic surface with four A_1 singularities, take $A \times \mathbb{P}^2$ and 6 general sections s_i which specialize to the intersection points of 4 lines l_i in the $\mathbb{P}^2$ over a_0. Blow up the sections s_i in $A \times \mathbb{P}^2$. The dual of the relative dualizing sheaf restricted to each fiber embeds the fiber away form a_0 as a D_3 in $\mathbb{P}^3$. The image of the fiber over a_0 is a cubic surface with the required singularities. There are 9 lines on the limit surface: the image of the 6 exceptional divisors and the image of the three lines joining the pairs of points that do not lie on an l_i. The descriptions of conics is similar.

Lemma 3.4 *Suppose C is a smooth rational curve in X^n contained in the locus of reducible conics Δ. Suppose C does not intersect the locus of non-reduced conics and $[C] \cdot \delta \geq 0$. Then C can be deformed away from Δ.*

Proof: C is contained in the smooth locus of Δ. Away from the locus of non-reduced conics Δ is a homogeneous variety and its tangent bundle is generated by global sections. Consequently, $N_{C/\Delta}$ is generated by global sections. Using the exact sequence

$$0 \to N_{C/\Delta} \to N_{C/X^n} \to \mathcal{O}_C(\Delta) \to 0$$

and the assumption that $[C] \cdot \delta \geq 0$, we conclude $h^1(C, N_{C/X^n}) = 0$ and that $H^0(C, N_{C/\Delta})$ does not surject onto $H^0(C, N_{C/X^n})$. Hence, the first order deformations of C are unobstructed and a general first order deformation of C does not lie in Δ. The lemma follows. $\square$

II. Degeneration of D_n to a scroll with a double line. The projection of a scroll $S_{1,2}$ or $S_{2,l}$, $l \leq 6$, from a general point on the plane of a conic in the surface has the same Hilbert polynomial as a Del Pezzo surface. We will show that these surfaces are limits of Del Pezzo surfaces. These scrolls can be further degenerated to more unbalanced scrolls.

The union of the double line with the fibers gives rise to a curve C of reducible conics in the Hilbert scheme X^n contained in the smooth locus of Δ. Since $\delta \cdot [C] = 8 - n$, we conclude by Lemma 3.4 that C can be deformed away from Δ. The resulting curve has the same class as a curve arising from a Del Pezzo surface. The surface in $\mathbb{P}^n$ spanned by the conics is a non-degenerate, irreducible surface of degree n ruled by reduced and generically irreducible conics. Since the dimension of scrolls with a choice of curve of conics sweeping the surface once is smaller than the dimension of the deformations of C, by Theorem 2.5 we conclude that the surface is D_n (recall $n < 8$).

Alternative construction when $n \leq 5$. For concreteness assume $n = 5$. Take $A \times \mathbb{P}^2$ and specialize 4 general sections s_i to the same point p on the central fiber. Blow up the total space at p, then along the proper transform of the sections s_i. Denote the total space of the resulting threefold by X. The central fiber is the union of F_1 with a $\mathbb{P}^2$ blown up at 4 points. Denote these two components of the central fiber by F and P, respectively. Take the linear system which restricts to $|e + 3f|$ on F and to $2L - \sum_{i=1}^4 E_i$ on P, where L is the line class on $\mathbb{P}^2$ and E_i are the exceptional divisors of the blow-ups. The linear series on the F_1 component is not complete, but must match the linear series on P. The latter contracts the surface to a double line. This constructs the desired degeneration of D_5.

III. Degeneration of D_n to a cone over an elliptic curve. Every surface degenerates to a cone over a hyperplane section (possibly with some embedded structure at the cone point) by taking the limit of a one parameter family of projective transformations fixing the hyperplane. Since the cone over an elliptic curve of degree n in $\mathbb{P}^{n-1}$ has the same Hilbert polynomial as D_n there are no embedded components in this case. By degenerating the elliptic curve which

is the base of the cone into an elliptic curve with rational tails, one obtains degenerations of D_n into an elliptic cone union rational cones.

IV. D_4 degenerates to the projection of a Veronese surface with a double line. Both D_4 and the Veronese surface with a double line are complete intersections of two quadric threefolds. Since a general complete intersection is a D_4, to obtain such a degeneration it suffices to specialize the quadrics.

To see that a Veronese surface with a double line is a complete intersection of quadric threefolds it suffices to observe that such a Veronese surface is given by the map

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2)$$

in projective coordinates, hence two quadric threefolds contain it.

A degeneration of D_n into D_4 union other surfaces meeting D_4 along lines further degenerates to a surface where a component is a Veronese with a double line.

V. Degeneration of D_n to D_{n-1} union a plane. In $A \times \mathbb{P}^2$ blow up $9 - n$ disjoint sections $p_i(a)$ of A in general position. In the fiber over a_0 blow up a general point q. The fibers away from a_0 are the blow-up of $\mathbb{P}^2$ at $9 - n$ points. The central fiber has two components: W, the blow-up of $\mathbb{P}^2$ at $10 - n$ points and Z, the exceptional divisor of the blow-up of q. Over the punctured disk $A^* = A - a_0$ the dual of the relative dualizing sheaf is a line bundle. One of its flat limits restricts to the anti-canonical bundle on W and to $\mathcal{O}_{\mathbb{P}^2}(1)$ on Z. This provides us with the desired family.

The limit of the lines are the lines that do not intersect $W \cap Z$. The conics on the general fiber correspond to lines going through p_i, conics passing through 4 of the points p_i or cubics double at one p_i and passing through 5 of the other p_j. We describe the limits of conics corresponding to the lines passing through p_1. The others are analogous. In the limit this curve of conics has two components. One component corresponds to lines passing through $p_1(0)$ on W. The other component consists of the union of the line l joining $p_1(0)$ and q and a line in $\mathbb{P}^2$ meeting l.

The limits of the hyperplane sections have three components. One component consists of elliptic curves of degree n on W in the class $3H - \sum_{i=1}^{9-n} E_{p_i}$. One component consists of conics on Z and rational curves of degree $n - 1$ meeting the conics twice on W. The last component corresponds to sections by hyperplanes that do not contain W or Z.

Similar constructions give examples of degenerations of D_n to D_{n-k} (with various singularities) union a rational cone of degree k. For example, (as B. Hassett pointed out) to obtain a degeneration of D_n to D_{n-2} with an A_1 singularity union a quadric cone with vertex at the singular point and having a common line with D_{n-2}, blow up the central fiber of $A \times \mathbb{P}^2$ at a general point, then blow up a general line in the exceptional divisor. Call the exceptional divisors of the blow-ups E_1 and E_2, respectively. Pick $9 - n$ general sections that specialize to the proper transform of the central fiber and blow them up. Denote the resulting three-fold by X. The linear series $|-K_X - 2E_1 - E_2|$, where K_X is the canonical bundle of X, gives the desired degeneration.

VI. Degenerations of D_n to a Veronese surface union a rational normal scroll meeting along a conic. The scroll must have degree $n-4<4$. These scrolls each have at least a one parameter family of conics (possibly reducible).

Consider a Veronese surface union a scroll $S_{0,1}$, $S_{1,1}$ or $S_{1,2}$. meeting along a conic. Choose 5, 3, 2 points on their common conics, respectively. Let C_1 be the curve of conics that contain all but one of the points on the scroll. Let C_2 be the curve of conics that contain the remaining point on the Veronese. This gives a reducible curve $C = C_1 \cup C_2$ in X^n in the class d_n. The normal bundle N_{C_i/X^n} is generated by global sections. This is clear for C_2 since it lies in a homogeneous locus and follows for C_1 by Lemma 3.4 after a simple specialization. The curve can be smoothed to an irreducible curve $\tilde{C}$ in the same class. By Theorem 2.5 the surface $\tilde{S}$ corresponding to $\tilde{C}$ must be D_n. Note that the total space of the family is singular at points that define C_i.

Alternative description when $n=5$. Choose 4 general sections $p_i(a)$ in $A \times \mathbb{P}^2$ that specialize to lie on a line l in the central fiber. Blow up $A \times \mathbb{P}^2$ along l, then blow up the proper transforms of $p_i(a)$. The general fiber is the blow up of $\mathbb{P}^2$ at 4 points. The central fiber is $\mathbb{P}^2$ union the blow-up of F_1 at 4 points where the two surfaces are joined along l. The dual of the relative dualizing sheaf is a line bundle away from the central fiber. To obtain a map that extends to the central fiber we have to twist by the plane in the central fiber. The limit restricts to $\mathcal{O}_{\mathbb{P}^2}(2)$ on $\mathbb{P}^2$ and to $\mathcal{O}_{F_1}\left(e+f-\sum_{i=1}^4 E_i\right)$ on the blow-up of F_1. The image is a Veronese surface union a plane. The total space of the image in $A \times \mathbb{P}^5$ has 5 singular points. They correspond to the intersection points of l with the fibers of F_1 that are blown down and with the curve in the class $e+2f$ passing through the p_i which is also blown down. The limits of the lines and conics are clear.

VII. Degenerations of D_n to the union of two scrolls sharing a conic. D_3's can specialize to the union of a plane and a quadric surface. D_4's can specialize to the union of a plane and $S_{1,2}$ or to the union of 2 quadric surfaces. D_5 can specialize to the union of a plane and $S_{2,2}$ or the union of the quadric surface and $S_{1,2}$. D_6 can specialize to the union of two $S_{1,2}$, a quadric surface and an $S_{2,2}$ or a plane and $S_{2,5}$. D_7 can specialize to $S_{1,2}$ union $S_{2,2}$, a quadric surface union $S_{2,3}$, a plane and $S_{2,4}$. Since further unbalanced scrolls are limits of balanced scrolls those also arise as limits.

Take the curve in X^n whose points correspond to incident line pairs one in each surface. In case one of the scrolls is $\mathbb{P}^2$ take the lines on $\mathbb{P}^2$ containing a fixed point on the common conic. This curve smooths. When each of the scrolls have a $k_i \geq 1$ parameter family of conics, the conics on each scroll passing through $k_i - 1$ points on the common conic give a different curve which can be smoothed.

In a general family arising in one of these ways, the limit of hyperplane sections have three components. Two of the components correspond to elliptic curves on S_{k_i,l_i} of degree $k_i + l_i + 2$ union $k_j + l_j - 2$ fibers on S_{k_j,l_j} meeting the elliptic curve. The third component corresponds to hyperplane sections by hyperplanes not containing either of the two components. Since for any choice

of points the curves can be smoothed in X^n, we can also conclude that every elliptic curve of degree $k_i + l_i + 2$ occurs as the limit of some family.

D_8**.** The arguments for the degenerations in I, III and V apply to D_8 verbatim. However, since the anti-canonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ is also a smooth degree 8 surface in $\mathbb{P}^8$ that contains a curve of conics in the class d_8, the arguments in II, VI and VII only show that one of the two surfaces degenerates to the potential limit. Ciro Ciliberto pointed out to us that it is possible to modify the alternative construction in II using a Cremona transformation $(3,3,3,1)$ to obtain a degeneration of D_8 to a scroll with a double line. In VI when we smooth the Veronese union $S_{2,2}$ we obtain D_8 since the limit surface does not contain two rulings by conics where a conic from one ruling meets every conic in the other ruling as a limit of $\mathbb{P}^1 \times \mathbb{P}^1$ should. In VII we note that the union of two $S_{2,2}$ does not smooth to D_8 since it does not contain a line meeting every conic as a limit of D_8 should.

Degenerations of the Veronese surface. Since the Veronese surface appears as a component of the limits of D_n, we mention the non-degenerate two component limits of it. They are the union of a plane and a cubic scroll where the cubic scroll meets the plane along the directrix or the union of two quadric cones that share a vertex and a common line. Both cases occur. The Veronese surface degenerates to a cone over a rational normal quartic. The latter is a further specialization. To obtain the former limit carry out the usual construction by blowing up $A \times \mathbb{P}^2$ at a point on the central fiber.

Since the surface spans $\mathbb{P}^5$ the components must be scrolls meeting along a line. They can have degrees 2, 2 or 1, 3. Since a cone over a twisted cubic is a limit of $S_{1,2}$, we can assume that the cubic surface is smooth. The cubic plane pair cannot be joined along a fiber line and the quadrics have to be both singular with a common vertex. The former cannot happen because any two such surfaces are projectively equivalent. The dimension of the locus of pairs of a cubic surface union a plane meeting it along a fiber is too large. To show that the quadrics have to be as described we only need to show that the union of two quadric cones that share a common line but have distinct vertices cannot be a limit of Veronese surfaces.

Each Veronese surface has a two-parameter family of conics, hence gives rise to a $\mathbb{P}^2$ in X^5. The flat limit of the conics has to be a surface in X^5 connected in codimension 1. If a pair of quadric cones with distinct vertices were a limit, then the two surfaces would need to share a curve of conics (possibly reducible). Since this is not the case, we conclude that the vertices have to coincide.

A forthcoming paper of Dan Avritzer should elucidate the enumerative geometry of the Veronese surface. We are thankful to him for conversations on the degenerations of the Veronese surface.

4 Examples of Counting Del Pezzo Surfaces

In this section we illustrate with a few examples how to use our knowledge of the degenerations of Del Pezzo surfaces to study their enumerative geometry.

Example 1: Counting cubic surfaces in $\mathbb{P}^4$. By Lemma 2.3 the dimension of the space of cubic surfaces in $\mathbb{P}^4$ is 23. We can ask for the number of D_3's containing r points and meeting $23-2r$ lines.

The number of cubic surfaces containing 3 points and meeting 17 lines. We outline how to see that there are 36 cubic surfaces satisfying the required incidences using degenerations. (See Figure 2.) Fix a hyperplane H in $\mathbb{P}^4$.

Step I. Specialize the three points and a line l_1 to H.

• Some cubic surfaces can lie in H. These surfaces must meet the 16 intersection points of H with the lines outside H. They must also contain the original 3 points. There is a unique D_3 containing the 19 points. It counts with multiplicity 3 for the choice of intersection point of the surface with l_1.

• If a cubic surface does not lie in H, then its hyperplane section must be contained in the plane P spanned by the three points in H, so the surface must contain $P \cap l_1$.

Step II. Specialize a second line l_2 to H. Some cubics can now lie in H. Such a cubic must meet 19 points—the 4 points in the plane P and the 15 points of intersection of H with the lines outside H. These points impose independent conditions on cubics, so there is a unique solution counted with multiplicity 3 for the choice of intersection with l_2.

Step VI. This pattern continues until we specialize 6 lines to H. After we specialize 6 lines, if the cubic does not lie in H, then the hyperplane section has to be the unique cubic curve in P passing through the 9 points in P. When we specialize the next line, either the cubic lies in H or it must break into P and a quadric meeting the rest of the lines. By a dimension count this is the first stage where reducible solutions occur.

Step VII. We are reduced to counting quadric surfaces meeting 10 lines and containing a conic in common with P. Further degeneration shows that there are 15 such quadrics. Briefly, specialize a line l_8 to H. Either the quadric lies in H or it contains the point of intersection of l_8 with P. If the quadric lies in H, then it must contain the intersection points of the 9 remaining lines with H. This uniquely determines the quadric. It counts with multiplicity two for the choice of intersection with l_8.

The same pattern continues until we have 5 lines remaining outside. Then the hyperplane section of the quadric is the unique conic passing through the 5 points in P. Once we specialize another line, the quadric has to either lie in H or break into a union of P and another plane having a common line with P and meeting the remaining 4 lines. The latter number is 3 by elementary Schubert calculus. We conclude that there are 36 cubic surfaces in $\mathbb{P}^4$ meeting 17 lines and containing 3 points. We will later verify the multiplicities (see §5.2).

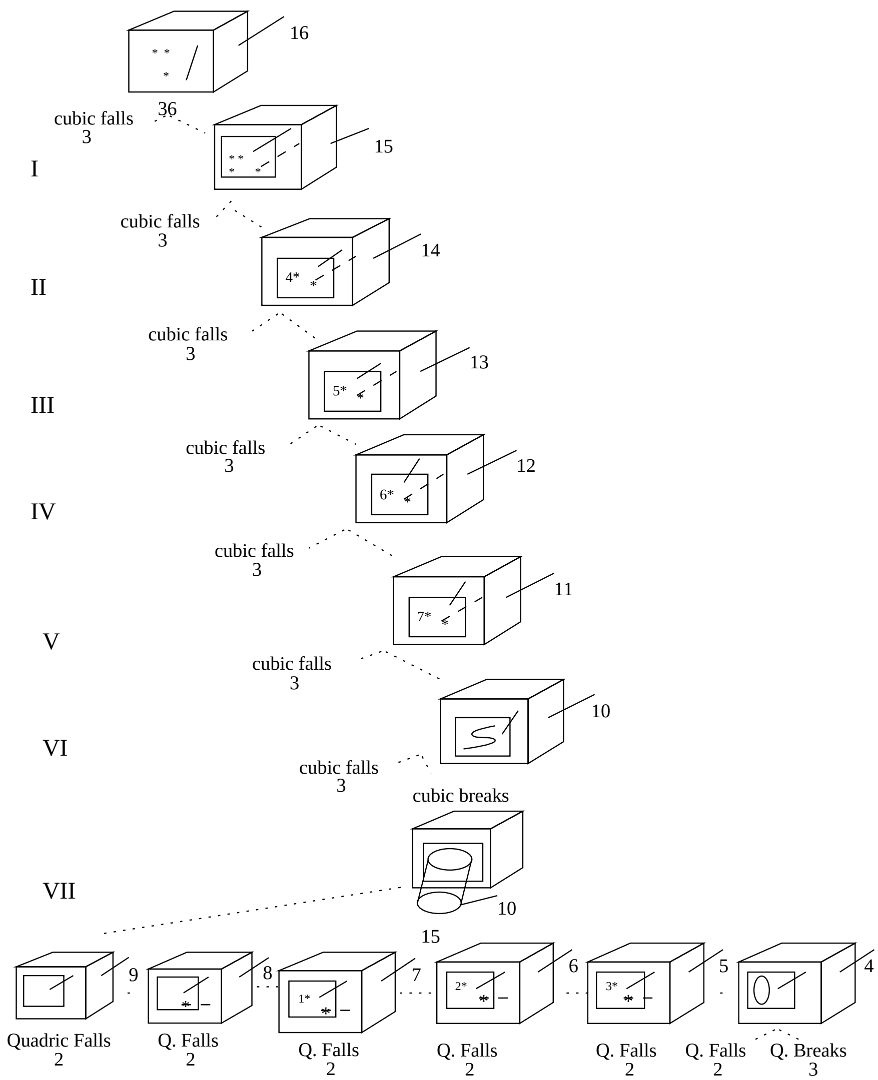

Figure 2: Cubic surfaces in $\mathbb{P}^4$ containing 3 points and meeting 17 lines (Example 1)

Example 2: Counting D_4's in $\mathbb{P}^4$. By Lemma 2.3 the dimension of the locus of D_4's in $\mathbb{P}^4$ is 26. We can ask for the number of D_4's containing r general points and meeting $26-2r$ general lines.

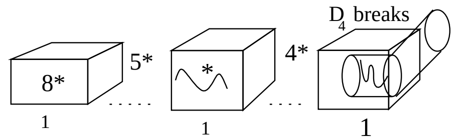

Figure 3: One D_4 containing 13 general points (Example 2A)

A. The number of D_4's in $\mathbb{P}^4$ containing 13 points. Specialize the points to a hyperplane H of $\mathbb{P}^4$. No reducible surfaces satisfy all the incidences until we specialize 9 points to H. After we specialize 8 points to H, the hyperplane section of a solution must be the unique elliptic quartic containing the 8 points. When we specialize a ninth point to H, Bezout's Theorem forces the surface to break into 2 quadric surfaces. The quadric surface Q in H is determined. There is a unique quadric in the $\mathbb{P}^3$ spanned by the 4 points not in H containing the intersection of Q with the $\mathbb{P}^3$ and the 4 points. After we verify the multiplicity claims, we can conclude that there is a unique D_4 containing 13 general points.

B. The number of D_4's in $\mathbb{P}^4$ containing 10 points and meeting 6 lines. (See Figure 4.) Fix a hyperplane H of $\mathbb{P}^4$.

Step I. Specialize 6 points and 3 lines $l_1,\ l_2,\ l_3$ to H. This is the first stage where reducible solutions occur: there can be the union of two quadric surfaces meeting along a conic. Of the three lines outside H, 3, 2, 1 or 0 of them might meet the quadric in H. The remaining 0, 1, 2 or 3 lines need to meet the quadric outside H. In each case there is a multiplicity of 8 for the choice of intersection points of the lines in H with the quadric in H. In two cases there is a combinatorial choice of 3 for which of the lines meet the quadric in H. The surfaces are uniquely determined.

Step II. If a solution is still irreducible, then its hyperplane section in H is an elliptic quartic curve C meeting the 6 points and the lines l_1, l_2, l_3. Specialize a line l_4 to H. The new reducible solutions must contain a curve C as described. We specialize three points and two lines l_1, l_2 to lie in a plane P in H. Either C passes through the intersection point of l_1 and l_2 or it must have a component in H. This component can either be a conic or a line. The residual component must be a conic or a twisted cubic, respectively. The number of these can be determined using the algorithm in §7 of [V]. Finally, of the lines outside H, 2, 1 or 0 of them might meet the quadric in H. Considering all the cases we see that there are 128 reducible surfaces at this stage.

Step III. If a solution is still irreducible, then its hyperplane section in H must be one of the 32 elliptic quartic curves containing 6 points and meeting 4 lines ([V] §8.3). We specialize a fifth line l_5 to lie in H. The Del Pezzo surfaces

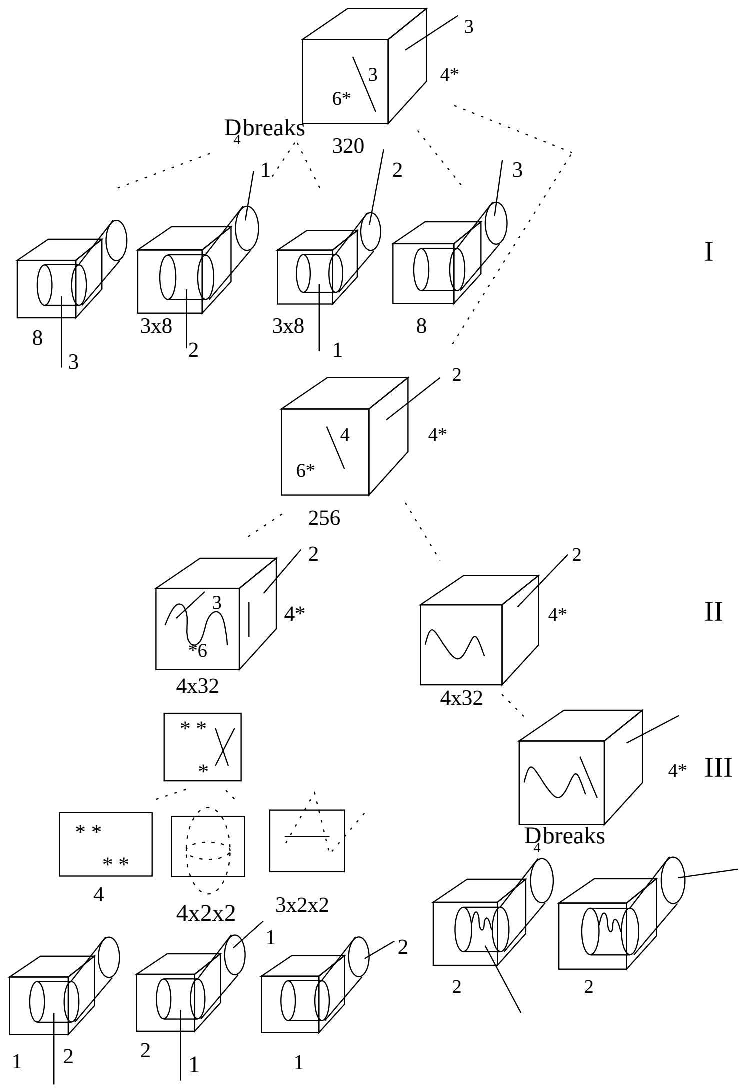

Figure 4: Counting D_4 surfaces containing 10 general points and meeting 6 general lines (Example 2B)

now have to break into a union of two quadrics. The sixth line can either meet the quadric in H or the quadric not lying in H. In each case there is a unique surface, appearing with multiplicity 2 for the choice of intersection of l_5 with the quadric in H. We conclude that there are 320 quartic Del Pezzo surfaces in $\mathbb{P}^4$ containing 10 general points and meeting 6 lines.

Example 3: Counting D_5's in $\mathbb{P}^5$ containing an elliptic quintic curve. As a final illustration of the type of enumerative problems one can hope to answer using degenerations, we find the number of D_5's in $\mathbb{P}^5$ containing an elliptic quintic curve, three points and meeting a plane. (See Figure 5.)

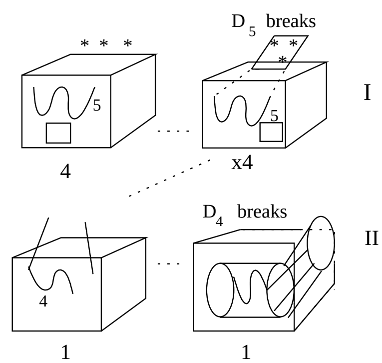

Figure 5: Counting D_5's containing a quintic elliptic curve, 3 points and meeting a plane (Example 3)

Step I. Specialize the plane P to the hyperplane H spanned by the elliptic quintic. The surface breaks into a union of D_4 and a plane Π. (By a dimension count it cannot break into a cubic scroll union a quadric surface. The other possibilities in Proposition 3.2 are either further degenerations of D_4, hence are excluded by a dimension count or do not contain any elliptic quintic curves—e.g. a Veronese surface or a quartic scroll.) We reduce the problem to counting D_4's containing an elliptic quintic C and a disjoint line l ($l = H \cap \Pi$). We get a multiplicity of 4 for the choice of intersection of P with any limit D_4.

Step II. Specialize the elliptic quintic to the union of an elliptic quartic and a general line. D_4 must become reducible by Bezout's Theorem since the hyperplane spanned by the elliptic quartic meets l. The surface must break into a union of quadrics and they are both uniquely determined. We conclude that there are four quintic Del Pezzo surfaces containing an elliptic quintic, three points and meeting a plane.

A non-example. Even when we impose only point conditions on surfaces D_n for $n > 4$, at each stage new and more complicated degenerations appear. One can make arbitrarily long lists of degenerations to count more cases until the dimension counts or the multiplicity calculations lose their rigor. We will

instead limit ourselves to simple enumerative problems. We give, however, an example to illustrate the complications due to curve conditions.

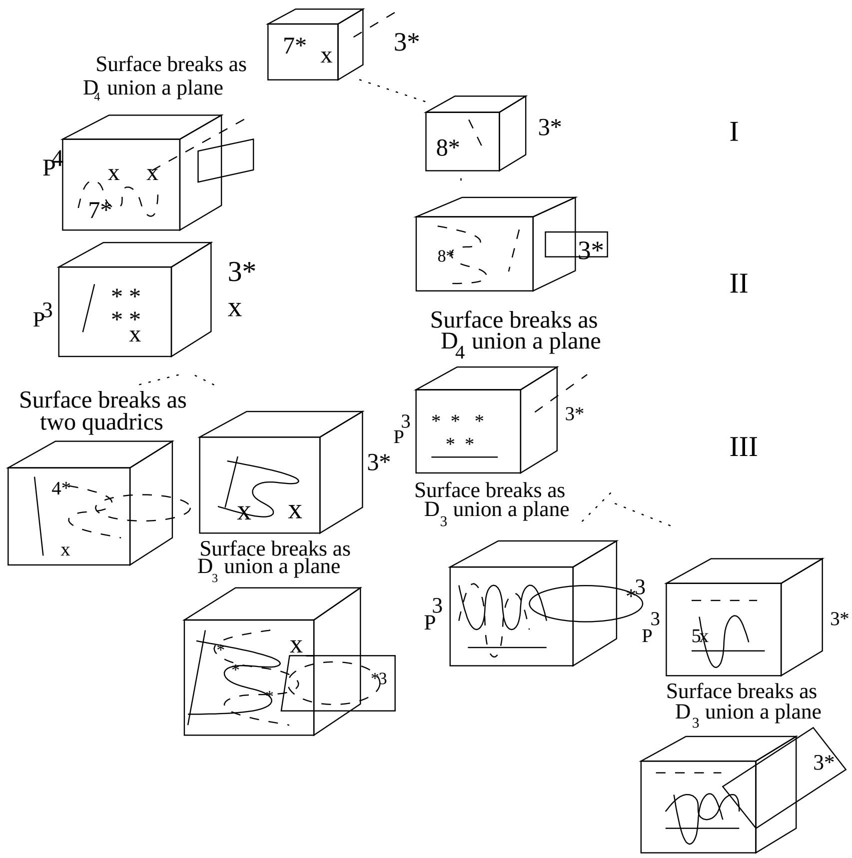

Figure 6: Degenerations of D_5's containing 11 points and meeting a line

The degenerations of D_5 in $\mathbb{P}^5$ that contain 11 points and meet a line l. (See Figure 6.) When we specialize 8 points to H, the surface can break into a plane P union a D_4. The D_4 must contain an elliptic quintic curve (the limit of the hyperplane sections of D_5) passing through the first 7 points (marked by *), the line $l_P = P \cap H$ and 2 other points (marked by x)—the 8th point and the point $l \cap H$. We specialize l_P, 4 of the 7 points (*) and one of the points (x) to a $\mathbb{P}^3$. If the surface does not break, then the hyperplane section is l_P union the twisted cubic C meeting it twice and containing 5 points (4 * and one x). When we specialize the last point (x), D_4 breaks into a cubic surface union a plane P'. The elliptic quintic must break into a union of a twisted cubic C' with a conic outside $\mathbb{P}^3$ meeting it twice since 3 of the points (*) are still outside the $\mathbb{P}^3$. Now we need to determine cubic surfaces containing the line l with the

twisted cubic C meeting it twice, containing a line l' which is the intersection of P' with the $\mathbb{P}^3$ and in addition containing a twisted cubic C' which meets l' twice and contains the 4 points (*) on C. The other cases are similar. As this example indicates when n gets larger, the curve conditions on D_n get very complicated making it very hard to continue the degenerations.

5 The Enumerative Geometry of D_n

In this section we carry out the dimension and multiplicity calculations necessary to justify calculations similar to ones in §4.

5.1 Dimension Counts

The building blocks. We calculate the dimension of relevant loci in the Hilbert scheme of surfaces.

Lemma 5.1 *The dimension D of the locus of surfaces S in $\mathbb{P}^N$ containing an irreducible, reduced conic in a fixed hyperplane H is as follows:*
1. If S is a scroll $S_{2,l}$, $l \geq 2$, then $D = N(l+4) - 3$.
2. If S is a scroll $S_{1,l}$, $1 \leq l \leq 2$, then $D = N(l+4) - 3$.
3. If S is the Veronese surface, then $D = 6N - 4$.

Lemma 5.2 *The dimension D of the locus of surfaces S in $\mathbb{P}^N$ containing a line l in a fixed hyperplane H is as follows:*
1. If S is a rational cone $S_{0,l}$, then $D = N(l+2) - 5$.
2. If S is a smooth Del Pezzo surface D_n, then $D = N(n+1) - n + 8$.
3. If S is a cone over an elliptic normal curve of degree k tangent to H everywhere along l, then $D = N(k+1) - 2$.

Lemma 5.3 *The dimension D of the locus of pairs (S, C) in $\mathbb{P}^N$ where S is a surface and C is a curve on it is as follows:*
1. When S is a Del Pezzo surface D_n and C is an elliptic curve of degree n (resp. $n+1$) , then $D = N(n+1) + 10$ (resp. $D = N(n+1) + 11$),
2. When S is a scroll $S_{k,l}$, $l-k \leq 2$, and C is an elliptic curve in a bisection class $2e + (l-k+2)f$, then $D = N(k+l+2) + 2k + 4 - \delta_{k,l}$
3. When S is a Del Pezzo surface D_n and C is a rational curve of degree $n-1$, then $D = N(n+1) + 9$.

Proof: To prove the lemmas consider maps from $\mathbb{P}^2$, F_{l-2}, F_{l-1} or the blow-up of $\mathbb{P}^2$ at $9-n$ points to $\mathbb{P}^N$, up to isomorphism, given by the linear series $|\mathcal{O}_{\mathbb{P}^2}(2)|$, $\left|\mathcal{O}_{F_{l-2}}(e+lf)\right|$, $\left|\mathcal{O}_{F_{l-1}}(e+lf)\right|$, $|-K|$ respectively. In the cases where the surface is required to contain a curve in a fixed hyperplane H, we assume that the map is given by $(s_0, \cdots, s_N)$, where s_0 corresponds to H. The lemmas follow from the cohomology calculations in §2 and §2 of [C]. □

Gluing. We now prove that gluing surfaces along projectively equivalent rational curves to form a tree imposes the expected number of conditions.

Notation. For a variety $X \in \mathbb{P}^N$, let $\mathcal{H}_X$ denote the locus in the Hilbert scheme parametrizing varieties projectively equivalent to X. Let $\mathrm{Rat}_d(X)$ be an irreducible subscheme of the scheme of rational normal curves of degree d on X. For a variety Z such that $[Z] \in \mathcal{H}_X$ let $\mathrm{Rat}_d(Z)$ denote the transform of $\mathrm{Rat}_d(X)$ under the projective linear transformation that takes Z to X. Let $(p_i^X)_{i=1}^s$, $0 \leq s \leq 3$, denote distinct points on a rational normal curve.

Lemma 5.4 *Let X_1 and X_2 be two varieties in $\mathbb{P}^N$. In the incidence correspondence $I :=$*

$$\left\{\left(Z_1, C_{Z_1}, (p_i^{Z_1})_{i=1}^s, Z_2, C_{Z_2}, (p_i^{Z_2})_{i=1}^s\right) : Z_j \in \mathcal{H}_{X_j}, C_{Z_j} \in \mathrm{Rat}_d(X_j), p_i^{Z_j} \in C_{Z_j}\right\}$$

the locus $C_{Z_1} = C_{Z_2}$ and $p_i^{Z_1} = p_i^{Z_2}$ for all $1 \leq i \leq s$ has codimension

$$N(d+1) + d - 3 + s.$$

Proof. Without loss of generality we can assume that $s = 0$. I maps to $\mathrm{Rat}_d(\mathbb{P}^N) \times \mathrm{Rat}_d(\mathbb{P}^N)$ by projection. The fibers are equivalent under the diagonal action of $\mathbb{P}GL(N+1)$. Since the locus of interest is the inverse image of the diagonal, the lemma follows. $\square$

Notation. Let H and Π denote two hyperplanes in $\mathbb{P}^N$. Let $\Sigma^j_{a_j}$ and $\Omega^i_{b_i}$ be collections of general linear subspaces of H and $\mathbb{P}^N$ of dimension a_j and b_i, respectively. Similarly, let $\Lambda^j_{a_j}$ and $\Gamma^{j'}_{a_{j'}}$ be collections of general linear spaces of $\mathbb{P}^N$ and Π, respectively. We will usually omit the dimension from the notation. We denote connected curves of arithmetic genus 1 by E and connected curves of arithemetic genus 0 by R. To denote their degree we append a number in parentheses.

Let $\mathcal{H}(\mathbb{P}^N, D_n)$ be the component of the Hilbert scheme whose general point corresponds to a smooth Del Pezzo surface D_n. Let $\mathcal{E}(\mathbb{P}^N, m)$ denote the component of the Hilbert scheme whose general point represents a smooth elliptic curve of degree m in $\mathbb{P}^N$. Let $\mathcal{HE}(\mathbb{P}^N, D_n, m)$ be the incidence correspondence of pairs

$$\left\{([D_n], [E(m)]) \in \mathcal{H}(\mathbb{P}^N, D_n) \times \mathcal{E}(\mathbb{P}^N, m) : E \subset D_n\right\}$$

where the elliptic curve E is a closed subscheme of the Del Pezzo surface D_n. Finally, let $\mathcal{U}(\mathbb{P}^N, D_n, m, I, J)$ denote the (I, J) pointed universal surface curve pair over $\mathcal{HE}(\mathbb{P}^N, D_n, m)$ defined by

$$\left\{\left(D_n, E, (q_i)_{i=1}^I, (p_j)_{j=1}^J\right) : p_j \in E \subset D_n, q_i \in D_n, (D_n, E) \in \mathcal{HE}(\mathbb{P}^N, D_n, m)\right\}$$

where p_j and q_i are points of the curve E and surface D_n, respectively.

The space of Del Pezzo surfaces. Let $\mathcal{D}_n(\mathbb{P}^N, I_1, I_2, J)$ denote the closure in $\mathcal{U}(\mathbb{P}^N, D_n, n, I_1 + I_2, J)$ of

$$\left\{\left(D_n, E, (q_i)_{i=1}^I, (p_j)_{j=1}^J\right) : E = D_n \cap H, q_i \subset \Omega^i, p_j \subset \Sigma^j \subset H\right\}$$

where D_n is a smooth Del Pezzo surface, E is its hyperplane section in H and the marked points are requied to lie in the designated linear spaces. This notation

is bad since many different possibilities are denoted by the same symbol. Since in any given enumerative problem the dimensions of the linear spaces will be predetermined we will use it as a shorthand.

Let $I_1 \cup I_2 = I$ and $J_1 \cup J_2 = J$ be two partitions.

The space of scroll pairs. Let $\mathcal{S}(\mathbb{P}^N, k_1, l_1, k_2, l_2, I_1, I_2, J_1, J_2)$ denote the closure in $\mathcal{U}(\mathbb{P}^N, D_n, n, I_1 + I_2, J_1 + J_2)$ of the locus

$$\begin{aligned}&\{\,(S_{k_1,l_1} \cup S_{k_2,l_2}, E\,(k_1+l_1+2) \cup F_1 \cup \cdots \cup F_{k_2+l_2-2}, q_i, p_j) : S_{k_1,l_1} \subset H \\ &\ E\,(k_1+l_1+2) \subset S_{k_1,l_1}, F_1 \cup \cdots \cup F_{k_2+l_2-2} \subset S_{k_2,l_2} \cap H, S_{k_1,l_1} \cap S_{k_2,l_2} = R(2) \\ &\ q_i \in \Omega^i \cap S_{k_r,l_r} \text{ for } i \in I_r,\ p_j \in \Sigma^j \cap E\,(k_1+l_1+2) \text{ for } j \in J_1, \\ &\ p_j \in \Sigma^j \cap (F_1 \cup \cdots \cup F_{k_2+l_2-2}) \text{ for } j \in J_2\}\end{aligned}$$

of pairs of scrolls meeting along a conic $R(2)$, where the scroll S_{k_1,l_1} is in H and contains an elliptic curve $E(k_1+l_1+2)$ and the other scroll is outside H and its intersection with H consists of the conic $R(2)$ and the fibers $F_1, \cdots, F_{k_2+l_2-2}$ and the marked points lie in the designated linear spaces. The elliptic curve E, needless to say, meets all the fibers F_i. We also assume that $k_2 + l_2 > 1$ to ensure that S_{k_2,l_2} can lie outside H.

The space of D_{n-1} union a plane. Let $\mathcal{PD}_{n-1}(\mathbb{P}^N, I_1, I_2, J)$ denote the closure in $\mathcal{U}(\mathbb{P}^N, D_n, n, I_1 + I_2, J)$ of the locus

$$\begin{aligned}&\{(D_{n-1} \cup \mathbb{P}^2, E(n), q_i, p_j) : D_{n-1} \cap \mathbb{P}^2 = R(1), E(n) \subset D_{n-1} \subset H, \\ &\ q_i \in \Omega^i \cap D_{n-1} \text{ for } i \in I_1, q_i \in \Omega^i \cap \mathbb{P}^2 \text{ for } i \in I_2, p_j \in \Sigma^j \cap E(n)\}\end{aligned}$$

where D_{n-1} is a smooth Del Pezzo surface in H, $E(n)$ is an elliptic curve on D_{n-1} and the marked points lie in the designated linear spaces.

Let $\mathcal{D}_{n-1}\mathcal{P}(\mathbb{P}^N, I_1, I_2, J_1, J_2)$ be the variant where $\mathbb{P}^2$ is in H and D_{n-1} is outside, the two meet in a line l and the marked curve is a conic in $\mathbb{P}^2$ meeting the hyperplane section of D_{n-1} residual to l in two points.

To generalize the discussion below to more cases one has to formulate similar loci corresponding to other reducible surfaces (possibly with more components) that occur in §3 like pairs of a Veronese and a scroll or an elliptic cone and a rational cone. In addition one has to allow for the surfaces outside to have tangencies with H along their common curves with the surfaces in H and the "limit curve" in the components in H to have correspondingly larger degree.

The space of marked D_n. Let $D_n(\mathbb{P}^N, m, I, J, J')$ denote the closure in $\mathcal{U}(\mathbb{P}^N, D_n, m, I, J + J')$ of the locus

$$\left\{\left(D_n, E, (q_i)_{i=1}^I, (p_j)_{j=1}^J, (o_{j'})_{j'=1}^{J'}\right) : q_i \in D_n \cap \Omega^i, p_j \in \Lambda^j \cap E, o_{j'} \in \Gamma^{j'} \cap E\right\}$$

where D_n is a smooth Del Pezzo surface, E is a degree m elliptic curve on it and the marked points lie in the designated linear spaces. The indices i are reserved for points on the surface, the indices j indicate points on E, but not in Π. Finally, the indices j' designate points that lie both on E and in Π.

Let $D_n(\mathbb{P}^N, m, I, J, J', \mathcal{O}(1))$ denote the analogous space, but where in addition the points $o_{j'}$ satisfy $\sum_{j'=1}^{J'} o_{j'} = \mathcal{O}_E(1)$ in the Picard group of the elliptic curve E.

Let $S_{k,l}(\mathbb{P}^N, k+l+2, I, J, J')$ and $S_{k,l}(\mathbb{P}^N, k+l+2, I, J, J', \mathcal{O}(1))$ denote the analogous space where D_n is replaced by a scroll $S_{k,l}$ and the elliptic curve has degree $k+l+2$.

Let $D_n(\mathbb{P}^N, r_1 + r_2 = m, I, J_1, J_2, J_1', J_2')$ denote the closure of the locus where E is a pair of rational curves of degrees r_1 and r_2 meeting at two points in $D_n(\mathbb{P}^N, m, I, J_1 + J_2, J_1' + J_2')$ and the conditions are distributed between the rational curves according to a partition.

We define the analogous locus $S_{k,l}(\mathbb{P}^N; r_1 + r_2 = k+l+2; I, J_1, J_2, J_1', J_2')$ for scrolls.

The divisors. The space $\mathcal{D}_n(\mathbb{P}^N, I, J)$ has a natural Cartier divisor

$$D_H(\mathbb{P}^N, I, J) := \{(D_n, E, q_i, p_j) \in \mathcal{D}_n(\mathbb{P}^N, I, J) : q_I \in H\}$$

defined by requiring one of the marked points on the surface to lie in H.

Let $D_\Pi(\mathbb{P}^N, D_n, I, J, J')$ on $D_n(\mathbb{P}^N, n+1, I, J, J')$ and $D_\Pi(\mathbb{P}^N, S_{k,l}, I, J, J')$ on $S_{k,l}(\mathbb{P}^N, E(k+l+2), I, J, J')$ defined by letting p_J lie in Π be analogous Cartier divisors.

There is a natural map ϕ from $\mathcal{U}(\mathbb{P}^N, D_n, m, I, J)$ to $\mathcal{H}(\mathbb{P}^N, D-n)$ given by projection.

Definition 5.5 *A divisor D of a subscheme A in $\mathcal{U}(\mathbb{P}^N, D_n, m, I, J)$ is called* **enumeratively relevant** *if the image of D under ϕ has codimension 1 in the image of A.*

One can list the components of $D_H(\mathbb{P}^N, I, J)$ that are enumeratively relevant Weil divisors whose general point corresponds to a surface that has reduced components and spans $\mathbb{P}^n$. Rather than give long lists, we will demonstrate how one produces such lists in examples similar to the ones in §4 and indicate what other type of behavior can occur when we require fewer point conditions.

The following lemma, which is a consequence of Kleiman's Transversality Theorem [Kl] (see Proposition 6.1 in [C]) allows us to carry out the dimension calculations when $I = 1$, $\Omega^j = H$ and $\Sigma^1 = \mathbb{P}^N$.

Lemma 5.6 *Let A be a reduced, irreducible subscheme of $\mathcal{D}_n(\mathbb{P}^N, I, J)$ and let p be one of the labeled points. Then there exists a Zariski open subset U of the dual projective space $\mathbb{P}^{N*}$ such that for all hyperplanes $[H] \in U$, the intersection $A \cap \{p \in H\}$ is either empty or reduced of dimension $\dim A - 1$.*

We describe the components of $D_H(\mathbb{P}^N, I, J)$ in $\mathcal{D}_n(\mathbb{P}^N, I, J)$ relevant to our examples.

$$1.\ \mathcal{D}_n(\mathbb{P}^N, I-1, J+1)$$

where $\Sigma^{J+1} := \Omega^I$ and the rest of the data is identical. To check that this locus is a divisor we use the above reduction. In that case the dimension for surfaces

does not change; however, the moduli for the points is one less. Since surfaces which are limits of D_n which do not contain a component in H lie in the closure of 1, we can now assume that all other limit surfaces have a component in H.

$$2.\ \mathcal{D}_n(\mathbb{P}^{N-1}, I, J \geq n)$$

where $q_I \in \Omega^I$ and $q_i \in H \cap \Omega^i,\ i < I$. Since the dimension of D_n in $\mathbb{P}^{N-1}$ is $n^2 + 10 + (n+1)(N-1-n)$, the choice for hyperplane section has dimension n and the points have the same moduli, this locus is a divisor. It is enumeratively relevant when $J \geq n$. When $J < n$, the hyperplane sections move in a positive dimensional linear series. The image of ϕ has positive dimensional fibers. We can now assume that the other components of D_H consist of reducible surfaces at least one component of which lies in H and one component lies outside H.

$$3.\ \mathcal{S}(\mathbb{P}^N, k_1, l_1, k_2, l_2, I_1, I_2, J_1, J_2),\ J_1 + J_2 \geq 8 - (k_2 + l_2 - 2) + \delta^{(0,1)}_{(k_1,l_1)},\ l_1 - k_1 \leq 2$$

where $\sum_i (k_i + l_i) = n,\ q_I \in S_{k_1,l_1}$ and the rest of the marked points are distributed between the two components according to some partition. By Lemmas 5.1, 5.3 and 5.4 and the constructions in §3 the locus 3 is a divisor. It is enumeratively relevant when $J \geq 8 - (k_2 + l_2 - 2)$ unless $(k_1, l_1) = (0, 1)$ in which case it is enumeratively relevant when $J \geq 9 - (k_2 + l_2 - 2)$.

$$4.\ \mathcal{PD}_{n-1}(\mathbb{P}^N, I_1, I_2, J \geq n > 3)$$

where $q_I \in D_{n-1}$ and the other points are distributed among the two components according to some partition. By Lemmas 5.2, 5.3 and 5.4 it is a divisor. It is enumeratively relevant when $J \geq n$.

5.2 The "Algorithm" for Counting D_3 and D_4

Now we outline the argument for counting D_3 and D_4 in greater detail. We begin with some lemmas necessary for multiplicity calculations.

Lemma 5.7 *Let E be an elliptic curve of degree $n+1$ on D_n in the class $-K + E_i$ then*

1. $H^1(D_n, N_{D_n/\mathbb{P}^N}) = 0$,
2. $H^1(D_n, N_{D_n/\mathbb{P}^N}(-E)) = 0$

Proof: Suppose $H^i(D_n, T_{\mathbb{P}^N} \otimes \mathcal{O}_{D_n}) = 0$ and $H^i(D_n, T_{\mathbb{P}^N} \otimes \mathcal{O}_{D_n}(-E)) = 0$ for $i > 0$. Then using the standard short exact sequence

$$0 \to T_{D_n} \to T_{\mathbb{P}^N} \otimes \mathcal{O}_{D_n} \to N_{D_n/P^N} \to 0$$

we conclude that $H^1(D_n, N_{D_n/\mathbb{P}^N}) \cong H^2(D_n, T_{D_n})$. The analogous statements hold when we twist the sheaves by $-E$. Tensoring the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to (\mathcal{O}_{\mathbb{P}^N}(1))^{N+1} \to T_{\mathbb{P}^N} \to 0$$

by $\mathcal{O}_{D_n}$ and $\mathcal{O}_{D_n}(-E)$ we conclude that

$$H^i(D_n, T_{\mathbb{P}^N} \otimes \mathcal{O}_{D_n}) = 0 \quad \text{and} \quad H^i(D_n, T_{\mathbb{P}^N} \otimes \mathcal{O}_{D_n}(-E)) = 0$$

for $i > 0$. Here we are using the fact that $H^2(D_n, \mathcal{O}_{D_n}(-E)) = 0$. This follows by Serre duality.

Finally, by Serre duality $H^2(D_n, T_{D_n}) = 0$ and $H^2(D_n, T_{D_n}(-E)) = 0$. The lemma follows. $\square$

Lemma 5.8 *Let D_3 be a smooth cubic surface. Let E be a curve in the class $-K$ on D_3. Then $H^1(D_3, N_{D_3,\mathbb{P}^N}(-E)) = 0$*

Proof: $N_{D_3/\mathbb{P}^N} \cong \mathcal{O}_{D_3}(-3K) \oplus (\mathcal{O}_{D_3}(-K))^{N-3}$. We twist the normal bundle by K. Since D_3 is a rational surface, $h^1(\mathcal{O}_{D_3}) = 0$; and $h^1(\mathcal{O}_{D_3}(-2K))$ vanishes by the Kodaira Vanishing Theorem. $\square$

Lemma 5.9 *Let Q be a smooth quadric surface. Let E be a curve in the class $\mathcal{O}_Q(2,2)$ on Q. Then $H^1(Q, N_{Q,\mathbb{P}^N}(-E)) = 0$*

Proof: Since $N_{Q/\mathbb{P}^N} \cong \mathcal{O}_Q(2,2) \oplus (\mathcal{O}_Q(1,1))^{N-3}$ the lemma follows from the cohomology of Q ([Ha] Ex. III.5.6). $\square$

Counting D_3. We now analyze the case of D_3.

Proposition 5.10 *Every enumeratively relevant component of $D_H(\mathbb{P}^N, I, J)$ in $\mathcal{D}_3(\mathbb{P}^N, I, J)$ is one of*

1. *$\mathcal{D}_3(\mathbb{P}^N, I-1, J+1)$ if $\sum_{j=1}^{J}(N-2-a_j) + N - 2 - b_I \leq 3N+3$,*
2. *$\mathcal{D}_3(\mathbb{P}^{N-1}, I, J \geq 3)$, or*
3. *$\mathcal{S}(\mathbb{P}^N, 0, 1, 1, 1, I_1, I \backslash I_1, J \geq 9)$.*

Each of these occurs with multiplicity 1.

Proof: This is a complete list of enumeratively relevant components. If the surface represented by a general point of a component is irreducible or contained in H, we already argued that cases 1 or 2 must hold. If the surface is reducible with at least one component in H and one component outside, then the component in H must be a plane. (Note that a trivial dimension count excludes the possibility that any component is a non-reduced plane.) The component outside is a possibly reducible quadric surface. Further specializing a smooth quadric strictly decreases the dimension and there are no new contributions from the choice of the limit of the hyperplane section in H. Finally, by Lemma 5.6 the surfaces do not intersect the intersection of two of the linear spaces Ω^i or Σ^j. We conclude that we have the complete list.

To prove that the components occur with multiplicity 1, we can assume that $I = 1$, $\Omega^I = \mathbb{P}^N$, and $\Sigma^j = H$. Using Lemma 5.6 repeatedly we conclude the proposition in general. Next by taking a general projection, we can assume that $N = 3$. Now the argument is an easy deformation argument. We will show that as we move p_I out of the plane there is a first order deformation of the limit

cubic which contains the deformation of the point. This suffices to conclude that the multiplicity is 1.

Let (X_0, X_1, X_2, X_3) be coordinates on $\mathbb{P}^3$. Suppose H is defined by $X_0 = 0$. We will write the deformation down for the case 3, as the others are even easier. We can assume that the plane and quadric pair is given by $X_0 Q(X_0, X_1, X_2, X_3)$. Suppose the limit point is $p = (0,0,0,1)$. We take the deformation $p_\epsilon = (\epsilon, 0, 0, 1)$ of the point away from X_0. There are already 9 points in H. Those define a unique cubic on X_0 given by $C(X_1, X_2, X_3)$. We can assume that p does not lie on C. We need to write a first order deformation of the surface which vanishes on C and contains p_ϵ to first order:

$$X_0 Q(X_0, X_1, X_2, X_3) - \epsilon \left[\frac{Q(0,0,0,1)}{C(0,0,1)} C(X_1, X_2, X_3) + X_0 Q' \right]$$

where Q' is any quadric. $\square$

Proposition 5.10 reduces the problem of counting D_3 to counting plane and quadric pairs meeting in a conic or counting D_3 in one lower dimensional projective space with incidence conditions on a hyperplane section E.

The techniques in [C] readily provide a solution of the first problem. The incidence conditions on E are simply incidence conditions on the plane. By Schubert calculus we can express these conditions in terms of multiples of Schubert cycles. The problem reduces to counting quadric surfaces whose span contains a plane satisfying Schubert conditions.

The second problem requires us to describe the enumeratively relevant components of $D_\Pi(\mathbb{P}^N, 3, I, J, J')$ in $D_3(\mathbb{P}^N, 3, I, J, J')$, $J' < 4$.

1. The components where E and the surface remain outside Π are of the form $D_3(\mathbb{P}^N, 3, I, J-1, J'+1)$ if $J' < 2$ and $D_3(\mathbb{P}^N, 3, I, J-1, J'+1, \mathcal{O}(1))$ if $J' = 2$. The $\mathcal{O}(1)$ condition on an elliptic cubic simply means that the intersection points of the curve with the linear spaces are collinear.

2. If Λ^J has dimension $N-2$ and one of the $\Gamma^{j'}$ also has dimension $N-2$, then E can meet $\Lambda^J \cap \Gamma^{j'}$. If the linear spaces do not have codimension two in $\mathbb{P}^N$, by Lemma 5.6 the locus where E meets their intersection is not a divisor.

3. If E lies in Π and the surface is irreducible and not in Π, then we get the component $D_3(\mathbb{P}^N, I, J+J')$ when $J' = 2$ or $D_3(\mathbb{P}^N, I, J+J', \mathcal{O}(1))$ when $J' = 3$ where the linear spaces meeting E now are the intersections of the linear spaces with Π.

4. If the curve breaks, but the surface does not break, then there is a line in Π and a conic not in Π meeting it twice. $D_3(\mathbb{P}^N, 1+2 = 3, I, J, J')$ is the divisor corresponding to this situation. This is an enumeratively relevant divisor when $J' = 3$. The locus where the conic is more special is a sublocus of this one and hence does not form a divisor.

There are two other possibilities: the surface can lie in Π or both the surface and E can break. Neither of these loci give divisors. The first locus is a specialization of 3. A simple dimension count excludes the second locus.

Claim: All these components occur with multiplicity 1 in $D_\Pi(\mathbb{P}^N, 3, I, J, J')$

Proof: By repeatedly applying Lemma 5.6, we can assume $I = 0$. There is a smooth morphism from $D_3(\mathbb{P}^N, 3, 0, J, J')$ to $\overline{M}_{1,J+J'}(\mathbb{P}^N, 3)$ given by sending the surface curve pair to the stable map which embeds the curve into $\mathbb{P}^N$. This morphism extends to a general point of the divisors listed above. We note that in case 2, the map has a contracted rational component. Both of the spaces are smooth of the expected dimension at the general points of the listed components and at their images. To show that the morphism is smooth, it suffices to check that the Zariski tangent space to the fibers have the expected dimension. The Zariski tangent space is given by $H^0(D_3, N_{D_3/\mathbb{P}^N}(-E))$. By Lemma 5.8 since the latter bundle does not have any h^1, it follows that the morphism is smooth. The claim is a consequence of Theorem 6.3 [V] since $D_\Pi(\mathbb{P}^N, 3, I, J, J')$ here is the pull-back of D_H in the notation there by the smooth morphism. □

Finally, observe that if the data $\tilde{I}, \tilde{J}$ differs from I, J by either including another $N-2$ dimensional linear space to I or by an $N-1$ dimensional linear space to J, then the number of the new surfaces is the degree of the surface in the case of I and the degree of the curve in the case of J times the number because of the choice of the point. See Proposition 6.2 [V].

Counting D_3 containing a conic or a line is similar, but easier. [V] (see §7.7-7.9) demonstrates how to count curves with the $\mathcal{O}(1)$ condition. Consequently, we can always count D_3 using the degeneration method. As a corollary, we can count pairs D_3 with an elliptic cubic curve which satisfies incidence conditions with linear spaces and a divisorial condition. Counting D_3 incident to linear spaces can be solved by classical means, however, at each stage this requires working out the cohomology ring of a new parameter space. In addition, it is considerably more difficult to count D_3 with curve conditions which satisfy incidences and divisorial conditions by classical methods.

Counting D_4. For simplicity we discuss how to count D_4 in $\mathbb{P}^N$ when at least 4 of the linear spaces are points, one has dimension $k \leq N-4$ and at least $5+k$ have dimension $N-4$ or less. We first study the case $N = 4$. The case $N > 4$ easily reduces to it.

Specialize 6 of the points to a hyperplane H keeping 4 of the remaining points outside H at all times. Specialize the rest of the conditions to H in order of increasing dimension. The enumeratively relevant components of $D_H(\mathbb{P}^4, I, J)$ are one of

1. $D_4(\mathbb{P}^4, I-1, J+1)$ or
2. $S(\mathbb{P}^N, 1, 1, 1, 1, I_1, I \backslash I_1, J \geq 8)$.

These occur with multiplicity 1.

Since the surface cannot lie in H, the first divisor covers all the possibilities where the surface is irreducible. By Proposition 3.2 and a simple inspection D_H does not contain any components whose general member parametrizes surfaces with more than 2 components. The only component of D_H where the components of the limit surface have degree 2 is the divisor 2. If the components have degrees 1 and 3, then the cubic spans $\mathbb{P}^4$. If the cubic spanned $\mathbb{P}^3$, it could meet at most 6 of the points, but the remaining 4 do not lie on a plane. By Lemmas 5.1 and 5.4 there cannot be a plane and cubic scroll pair either.

To see that they occur with multiplicity 1 we argue as in the case of D_3 by constructing a suitable first order deformation. Since case 1 is easier, we just write the explicit deformation in case 2. Without loss of generality we can take the the quadrics given by $X_0X_1, X_0^2+X_1^2+X_2^2+X_3^2+X_4^2$. Let H be $X_0=0$. The limit hyperplane section is an elliptic quartic E, so it is the intersection in $X_0=0$ of $X_1^2+X_2^2+X_3^2+X_4^2$ with a quadric Q. We can assume $I=1$ and that the limit point is $(0,i,0,0,1)$. We deform the point away from X_0 by considering $(\epsilon,i,0,0,1)$. We need a deformation which vanishes on E and contains the deformation of the point to first order:

$$X_0X_1-\epsilon\left[\frac{i}{Q(i,0,0,1)}Q(X_1,X_2,X_3,X_4)+X_0L_1(X_0,X_1,X_2,X_3,X_4)\right]$$
$$X_0^2+X_1^2+X_2^2+X_3^2+X_4^2+\epsilon X_0L_2(X_0,X_1,X_2,X_3,X_4)$$

where L_i are linear forms.

We thus, reduce the problem to counting pairs of quadric surfaces meeting in a conic where the one in H contains an elliptic quartic E satisfying incidences. This problem requires us to describe the enumeratively relevant components of $D_\Pi(\mathbb{P}^3;1,1;I,J,J')$ in $\mathcal{S}(\mathbb{P}^3;1,1;I,J,J')$. We specialize the conditions on E onto a plane by bringing 3 of the 6 points onto the plane keeping 3 of the points outside and then bringing the rest in order of increasing dimension. There are at least 8 linear spaces meeting E and at least 6 of them are points. When we specialize Γ^J

1. E can meet Π along Γ^J. We get $\mathcal{S}(\mathbb{P}^3;1,1;I,J-1,J'+1)$.

2. If one of the linear spaces in Π is already a line and the condition we specialize is a line, then E can meet their intersection point. This gives rise to $\mathcal{S}(\mathbb{P}^3;1,1;I,J-1,J')$.

3. E can break into a conic in Π and a conic outside meeting it twice. This gives rise to $S_{1,1}(\mathbb{P}^3;2+2=4,I,J,J')$. In this case the enumerative problem becomes trivial because the conditions have to split between Π and the plane spanned by the points outside Π. We are reduced to counting quadrics subject to point conditions.

4. The curve can break into a line in Π and a twisted cubic meeting it twice. This gives rise to $S_{1,1}(\mathbb{P}^3;1+3=4,I,J,J')$. There are a finite number of these since the curve has at least 2 more conditions other than the 6 points ([V]). The curve imposes 8 linear conditions on quadric surfaces, so this case is also very easy to count.

This is a complete list because an easy dimension count shows that there are no components of D_Π where the curve has more than two components and the point conditions imposed on the quadric preclude the possibility of its breaking into two planes. In particular, the limit of E cannot be the union of a line with an elliptic cubic.

Claim: The divisors above occur with multiplicity 1.

Proof: There is a rational morphism from $S_{1,1}(\mathbb{P}^3,4,I,J,J')$ to $\overline{M}_{1,J+J'}(\mathbb{P}^3,4)$ by sending the marked elliptic curve to the stable map which embeds it in $\mathbb{P}^3$. This map is well defined on an open set of each of the divisors and it gives a

smooth morphism. The proof is identical to the case of D_3 except instead of using Lemma 5.8, we use Lemma 5.9. □

Finally, to reduce the more general case to the case $N = 4$ specialize the conditions to a hyperplane containing the 4 points. Suppose by induction that we can count D_4 in $\mathbb{P}^{N-1}$ satisfying the analogous conditions and all the degenerations of that case. If the surface lies in the hyperplane at any stage, then we are reduced to a subcase of the case $N - 1$. If the surface breaks, the argument for $N = 4$ case shows that it must be two quadrics and the same analysis applies to the quadric in H.

Remark: For concreteness let us assume $N = 4$. Almost the same argument holds if we required only 8 of the linear spaces to be points. In this case the surface can break into a cubic scroll union a plane in H, but these numbers can be determined using [C]. When exactly 8 of the linear spaces are points, there are cases when we have to count cubic scrolls meeting a line twice. Although this problem is not explicitly addressed in [C], a simple modification of the argument for counting twisted cubics meeting a line twice ([V] §7.5) works.

If only 7 of the linear spaces are points, provided we specialize all the point but one to H first, the only new limit is D_3 union a plane. These can be counted but requires studying the degenerations of D_3 with an elliptic quartic on it. This case is marginally harder than counting pairs D_3 and an elliptic cubic.

If we relax the conditions a little further, we open Pandora's box. Singular surfaces appear in the limit and they can have tangencies with H. Identifying the divisors and their multiplicities becomes difficult. Unlike the simple examples here the multiplicities are not in general 1.

6 Gromov-Witten Invariants of X^N

In this section we study the relation between the Gromov-Witten invariants of X^N and the enumerative geometry of D_n. We prove that when $n = 3$ and in many cases when $n = 4$, the Gromov-Witten invariants involving incidence to linear spaces are enumerative. However, when $n \geq 5$, most are not enumerative.

Gromov-Witten Invariants. The Kontsevich spaces of stable maps possess a virtual fundamental class $[\overline{M}_{0,m}(X,\beta)]^{\mathrm{virt}}$ of the expected dimension

$$\dim X - K_X \cdot \beta + m - 3.$$

They are equipped with m evaluation morphisms $\rho_1, \cdots, \rho_m$ to X, where the i-th evaluation morphism takes the point $[C, p_1, \cdots, p_m, \mu]$ to the point $\mu(p_i)$ of X. Given classes $\gamma_1, \cdots, \gamma_m$ in the Chow ring A^*X of X, we obtain a class

$$\rho_1^*(\gamma_1) \cup \cdots \cup \rho_m^*(\gamma_m)$$

in $\overline{M}_{0,m}(X,\beta)$. We can evaluate its homogeneous component of top dimension on $[\overline{M}_{0,m}(X,\beta)]^{\mathrm{virt}}$ to obtain a number $I_\beta(\gamma_1, \cdots, \gamma_m)$ called the *Gromov-Witten invariant.* Explicitly,

$$I_\beta(\gamma_1, \cdots, \gamma_m) = \int_{[\overline{M}_{0,m}(X,\beta)]^{\mathrm{virt}}} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_m^*(\gamma_m).$$

Notation. Let $\Gamma_{\Lambda^a} \subset X^N$ denote the variety of conics in $\mathbb{P}^N$ incident to a linear space Λ^a of dimension a. Let γ_a denote the cohomology class of Γ_{Λ^a}.

Definition 6.1 *We call a Gromov-Witten invariant $I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m})$ of X^N* **enumerative** *if for a general set of linear spaces Λ^{a_i} the only stable maps $(C, p_1, \cdots, p_m; \mu)$ in $\overline{M}_{0,m}(X^N, d_n)$ with $\mu_*[C] = d_n$ and $\mu(p_i) \in \Gamma_{\lambda^{a_i}}$ are injective maps from irreducible source curves whose images coincide with a curve of conics on a smooth D_n. We call these maps* **enumerative maps**.

Remark. $I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m})$ is non-zero only when

$$\sum_{i=1}^{m}(N - 1 - a_i) = N(n+1) - n + 10 + m.$$

We always assume that this equality holds and that $a_i < N - 2$.

We say a stable map μ to X^N *sweeps out a variety* $V \subset \mathbb{P}^N$ if the set-theoretic image of the projection of the universal conic over the image of μ to $\mathbb{P}^N$ is V. If μ restricted to an irreducible component C_i of the domain curve C sweeps out an irreducible variety $V \subset \mathbb{P}^N$, we say that $(C_i, \mu_{|C_i})$ *sweeps out V k times* if the projection from the universal conic is a generically finite morphism of degree k.

If the obstructions for a stable map vanish, then the virtual fundamental class coincides with the usual one. Lemma 1.1 in [Ga], which we reproduce for the reader's convenience, states a local version.

Lemma 6.2 *If $h^1(C, \mu^* T_X) = 0$ for $(C, p_1, \cdots, p_m; \mu) \in \overline{M}_{0,m}(X, \beta)$, then $(C, p_1, \cdots, p_m; \mu)$ lies in a unique component Z of $\overline{M}_{0,m}(X, \beta)$ of dimension equal to the virtual dimension. Moreover,*

$$[\overline{M}_{0,m}(X,\beta)]^{\mathrm{virt}} = [Z] + R$$

where R is a cycle supported on the union of the components other than Z.

Since the normal bundle of a curve C in X^N corresponding to a fixed conic class on a smooth D_n is generated by global sections, the standard exact sequence

$$0 \to T_C \to \mu^* T_{X^N} \to N_\mu \to 0$$

implies that for an enumerative map $h^1(C, \mu^* T_X) = 0$. By Lemma 6.2 an enumerative Gromov-Witten invariant of X^N is equal to the ordinary scheme-theoretic intersection of the cycles $\rho_i^* \Gamma_{\Lambda^{a_i}}$ on the component of enumerative maps, where Λ^{a_i} are general linear spaces. An enumerative map has no automorphisms, so the cycles intersect at smooth points of this component. Since by Kleiman's Transversality Theorem they intersect transversely, we conclude

Proposition 6.3 *Let $\#D_n(\gamma_{a_1}, \cdots, \gamma_{a_m})$ be the number of Del Pezzo surfaces incident to general linear spaces of dimension a_i. Let R_n denote the number of distinct conic classes on D_n. Then, for an enumerative Gromov-Witten invariant of X^N we have the equality*

$$I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m}) = R_n \cdot \#D_n(\gamma_{a_1}, \cdots, \gamma_{a_m})$$

Remark. Lemma 2.2 implies that $R_3 = 27, R_4 = 10, R_5 = 5, R_6 = 3$, $R_7 = 2$ and $R_8 = 1$.

Non-enumerative I_{d_n} The Gromov-Witten invariants $I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m})$ are in general not enumerative. The following proposition constructs non-enumerative examples for $5 \leq n \leq 7$. When $n = 8$, the invariants are not only non-enumerative, but the conics that sweep a D_8 and the conics that sweep the anti-canonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ both contribute to them. Hence, the invariants are not well-suited for enumerative calculations.

Proposition 6.4 *The Gromov-Witten invariant $I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m})$ of X^n is not enumerative when*

1. *$n = 5$ and $\sum_{i=1}^m (2 - a_i) \leq 16$,*
2. *$n = 6$ and there exists a partition of the numbers a_i into $(b_s)_{s=1}^S$ and $(c_t)_{t=1}^T$ such that*

$$\sum_{s=1}^S (3 - b_s) \leq 12,\ \sum_{t=1}^T (2 - c_t) \leq 10,\ \sum_{s=1}^S (4 - b_s) = 36,\ \sum_{t=1}^T (4 - c_t) = 10,$$

3. *$n = 7$ and $\sum_{i=1}^m (3 - a_i) \leq 30$. If equality holds we also assume that at least one $a_i < 3$ is not equal to 1.*

Proof: For each case we need to construct a stable map to X^N that satisfies the incidences, but does not sweep out a smooth D_n. We already encountered the additional components of $\overline{M}_{0,m}(X^N, d_n)$ in the proof of Proposition 3.2. Take a smooth Del Pezzo surface D_{n-k}, $n - k \geq 3$, and a rational cone of degree k containing a line l of D_{n-k} and meeting a different line l' of D_{n-k} incident to l at its vertex. Denote this surface by $R(k, n)$ (see Figure 7). The connected curve of conics corresponding to the conics in the class $[l] + [l']$ on D_{n-k} and the union of l' with the lines on the cone has class d_n in X^N. The dimension lemmas in §5 imply that the dimension of $R(k, n)$ is $N(n+1) - n + 10 + k - 2$. When $k > 1$, this is at least the dimension of D_n. Under the hypotheses of the proposition it is easy to see that there are $R(2, 5)$, $R(2, 6)$ and $R(3, 7)$ meeting a general set of Λ^{a_i} in $\mathbb{P}^5, \mathbb{P}^6$ and $\mathbb{P}^7$ in the three cases, respectively. The same construction also provides non-enumerative examples of I_{d_n} when $N > n$. $\square$

Theorem 6.5 *The Gromov-Witten invariant $I_{d_n}(\gamma_{a_1}, \cdots, \gamma_{a_m})$ of X^N is enumerative when*

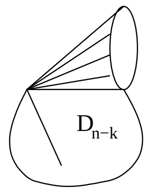

Figure 7: The surface $R(k, n)$.

1. $n = 3$ *or*

2. $n = 4$ *and* $\sum_{i=1}^{m}(N-3-a_i) > 4(N-3)$.

Proof: Let $(C, p_1, \cdots, p_m; \mu)$ be a stable map to X^N in the class d_n such that $\mu(p_i) \in \Gamma_{\Lambda^{a_i}}$ for general linear spaces Λ^{a_i}. Let V be the variety swept out by μ. Then V must meet the linear spaces Λ^{a_i}. The strategy of the proof is to use Theorem 2.5 to prove that V is a smooth D_n under our hypotheses.

Claim 1: The variety swept out by a connected curve in the class d_n spans at most $\mathbb{P}^n$.

If the curve in X^N is connected, then the threefold in $\mathbb{P}^N$ swept by the planes of the conics is connected in codimension 1. Since its degree is bounded by $n-2$, its span (which contains V) can be at most $\mathbb{P}^n$.

Claim 2: V is a variety of pure dimension 2.

If C is an irreducible curve in X^N, the variety it sweeps in $\mathbb{P}^N$ does not have to be of pure dimension 2. It can be of pure dimension 1 or it can have a component of dimension 1 and a component of dimension 2.

If a non-constant family in X^N sweeps a curve, then the curve is a line l and the conics are non-reduced conics whose set-theoretic support is l. Such a component sweeps a surface of degree 0. Since the degree of the surface swept by the conics in the class d_n is non-zero, V must contain a surface component. Since the image of μ is connected if it contains a component of the type just described, the line l must lie on a surface component.

If an irreducible curve of conics sweeps out a variety which has a dimension 1 and a dimension 2 component, then the conics consist of the lines on a cone union a line l meeting the cone at its vertex. We refer to lines like l as *needles.* (See Figure 8).

If V does not have pure dimension 2, then it must contain a needle. Since the image of μ is connected, if the needle is not contained in a surface component, then every component of the image of μ must have the same line as a needle. A stable map onto such a curve cannot be in the class d_n. The class of a stable map which sweeps a rational cone of degree r with a needle k times is $-kr\ a + kr\ b$. Hence, V has pure dimension 2.

Claim 3: If $n = 3$ (resp. 4), V is an irreducible surface of degree 3 (resp. 4).

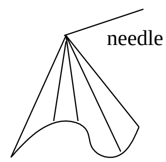

Figure 8: Needles.

Since meeting linear spaces in general position impose independent conditions, we can check this claim by a naive dimension count. V has degree at most n and spans at most $\mathbb{P}^n$.

When $n = 3$, the linear spaces impose $19 + 4(N - 3)$ conditions on V. If V has degree less than 3, then it is either a plane or a quadric surface, hence V has dimension at most $9 + 4(N - 3)$. We conclude that V has degree 3. This concludes the proof when $n = 3$ because any surface of degree 3 in $\mathbb{P}^3$ is a specialization of smooth D_3 surfaces. The variety of singular cubic surfaces has codimension 1 in the space of cubic surfaces. If the linear spaces are general, the only cubic surfaces that satisfy all the incidences will be smooth surfaces. We conclude that the image of μ must contain a curve of conics on D_3, but then the image must coincide with it. Also observe that μ cannot have any contracted components, since otherwise V would have to meet Λ^{a_i} and Λ^{a_j} for some i, j along the same conic. Dimension considerations exclude this possibility.

When $n = 4$, there are more cases to consider. It suffices to carry out the dimension counts when $N = 4$. Since the dimension of cubic surfaces in $\mathbb{P}^4$ is bounded by 23, the degree of V must be 4.

A surface swept by a connected curve in X^N is connected in codimension 1 by lines or conics except when it contains cones meeting only along their vertices.

The dimension of four-tuples of planes in $\mathbb{P}^4$ or triples of a quadric surface and two planes in $\mathbb{P}^4$ is bounded by 25. Hence, if V is reducible, then it is either the union of two quadrics or the union of a cubic surface and a plane. If the quadrics share a common line or conic or if one of them is a cone, then their dimension is strictly less than 26. The choice of pairs of a plane and a cubic that spans $\mathbb{P}^4$ is bounded by 24. If the cubic spans only $\mathbb{P}^3$ and is ruled by lines, it is either the projection of a cubic scroll or a cone over an elliptic curve. In either case the dimension of the choice of pairs of such a cubic and a plane is bounded by 23 (§5.1). If the cubic is not ruled by lines, then it contains only a one parameter family of conics and the plane must meet it along a line or a conic. By §5.1 the dimension of these pairs is also strictly less than 26. We conclude that V is irreducible.

Theorem 2.5 classifies irreducible quartic surfaces that span $\mathbb{P}^4$. The projection of a scroll of degree 4 in $\mathbb{P}^5$ and of a Veronese surface have dimensions bounded by 23 and 16, respectively. The other possibilities are degenerations of D_4, hence have dimension strictly smaller than 26. Finally by the assumption

that $\sum_{i=1}^m (N-3-a_i) > 4(N-3)$, no $\mathbb{P}^3$ meets all the linear spaces Λ^{a_i}. We conclude that V must span $\mathbb{P}^4$. This completes the proof that V is a smooth D_4. The rest of the argument is identical to the previous case. $\square$

Remark: When $n = 4$, can we remove the assumption on a_i? Any counterexample must arise from an irreducible quartic surface in $\mathbb{P}^3$. Suppose we take a quartic surface in $\mathbb{P}^3$ with a double line. The dimension of such surfaces is 25. The conics that are residual to the double line l in the pencil of planes containing it give us a curve C in X^3. The class of C is $2a + b$, not d_4. Suppose now we choose a more special quartic surface S so that C contains a non-reduced conic whose set theoretic support is a line m. Take the curve C' of non-reduced conics whose set theoretic supports are m, but whose planes rotate once about m in $\mathbb{P}^3$. The union of C and C' now are in the class d_4. The dimension of surfaces S in $\mathbb{P}^3$ with the required property is 22. We conclude that Theorem 6.5 is the sharpest we can hope for when $n = 4$.

The table below gives examples of Gromov-Witten invariants of X^N we can calculate using Theorem 6.5, Proposition 6.3 and the degeneration method in §5. We use the short-hand $I_{d_n}(a_1^{r_1}, \cdots, a_k^{r_k})$ to denote the Gromov-Witten invariant of X^N in the class d_n incident to r_i cycles of conics meeting linear spaces of dimension a_i.

$N=3$	$I_{d_3}(0^{19}) = 27$	$N=4$	$I_{d_4}(0^{13}) = 10$
$N=4$	$I_{d_3}(0^4, 1^{15}) = 27$	$''$	$I_{d_4}(0^{12}, 1^2) = 40$
$''$	$I_{d_3}(0^3, 1^{17}) = 972$	$''$	$I_{d_4}(0^{11}, 1^4) = 320$
$''$	$I_{d_3}(0^2, 1^{19}) = 21303$	$''$	$I_{d_4}(0^{10}, 1^6) = 3200$
$N=5$	$I_{d_3}(0^2, 1^4, 2^{13}) = 54$	$''$	$I_{d_4}(0^9, 1^8) = 33280$
$''$	$I_{d_3}(0^2, 1^3, 2^{15}) = 1863$	N=5	$I_{d_4}(0^4, 1^9, 2) = 240$

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